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Mann iteration with errors for strictly pseudo-contractive mappings.

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Abstract

***It is well known that any fixed point of a Lipschitzian strictly pseudo-contractive self mapping of a nonempty closed convex and bounded subset K of a Banach space X is unique* [6] *and may be norm approximated by an iterative procedure. In this paper, we show that Mann iteration with errors can be used to approximate the fixed points of strictly pseudocontractive mappings. Our result extend the corresponding result obtained by Liu* [6].**

Keywords: Strictly pseudocontractive mapping, Mann iterative process with errors.

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1.0 Preliminaries

The Mann iteration procedure has been introduced as a viable method to approximate the fixed points of a contractive operator when the Picard iteration does not converge [4].

Let X be a Banach space, let T: $X \to X$ be a map. Suppose $x_0 \in X$. The Mann iteration [6] is given by:

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, Tx_n,\ n \geq 0 \tag{1.1}$$

where the sequence $(\alpha_n)_n \subset (0,1)$, $\lim \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The Mann iteration with errors ([3], [5]) is defined as sequence (x_n) where

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, Tx_n + e_n \tag{1.2}$$

$(e_n) \subset X$ *satisfy* $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *and* (α_n) is the same as in (1.1). When $e_n = 0$ for all n, then we have the usual Mann iteration procedure in (1.1).

***Definition* 1 1** ([6])

Let X be a Banach space. A mapping T is said to be strictly pseudocontractive if there exists a number t >1 such that the inequality $\|x-y\| \leq \|(1+r)\ (x-y) - rt\,(Tx-Ty)\|$ holds for all x, y ε $D(T)$ and $r > 0$

***Definition* 1.2.** ([2] and [6]).

A mapping T is said to be strongly accretive if there exists a positive number k such that for each $x, y \in D(T)$ there is $j \in J\,(x - y)$ such that

$$\langle Tx-Ty,\ j \rangle \geq k\,\|x-y\|^2 \tag{1.3}$$

where J is the normalized duality mapping from X to 2^{x^*} given by

$$Jx = \left\{ f^* \in X^* : \left\| f^* \right\|^2 = \|x\|^2 = \mathrm{Re}\left\langle x, F^* \right\rangle \right\}.$$

The inequality (1.3) implies the inequality $\|x - y\| \le \|x - y + r[(T - kI)x - (T - kI)y]\|$, k ∈ (0,1).

***Definition* 1.3** ([6, L emma1]).

Let X be a Banach Space, K a subset of X, and $T: K \to K$. Then T is a strictly pseudocontractive mapping if and only if I - T is a strongly accretive mapping i.e. the inequality.

$$\|x - y\| \le \|x - y + r[(I - T - kI)x - (I - T - kI)y]\| \qquad (1.4)$$

holds for any $x, y \in k$ and $r > 0$, $k = \frac{t-1}{t}$ and I is the identity map.

***Definition* 1.4**

Let X be a Banach Space. Let K be a non empty closed, convex subset of a Banach space X and $T : K \to K$. T is Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\| \qquad (1.5)$$

for all $x, y \in K$. Of course, we are most interested in the case where $L \ge 1$.

The following lemma can be found in [10]

***Lemma* 1.5** ([10]).

Let a_n be a nonnegative sequence that satisfies the inequality $a_{n+1} \le (1 - \lambda_n)a_n + \sigma_n$

where $\lambda_n \in (0,1)$ for each $n \in N$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = \in_n \lambda_n$, $\lim_{n \to \infty} \in_n = 0$. Then $\lim_{n \to \infty} a_n = 0$

The Mann iteration with errors have been used by several authors to approximate the fixed point of Lipschitzian strictly pseudocontractive maps defined on uniformly smooth Banach space. For example, see [3] and [5].

The purpose of this paper is to show that the Mann iteration with errors converges strongly to the unique fixed point of a Lipschitzian and strictly pseudocontractive mapping. Chidume [2] considered the Mann iterative process for strongly pseudocontractive maps defined on an Hilbert space. Ding [3] improve on Chidume’s result by considering the Mann iteration with errors, though for locally strictly pseudocontractive maps, which is not as general as strongly pseudocontractive maps. However, Baeke et al [1] improved on the result by considering the Mann iteration with errors for strongly pseudocontractive maps defined on an Hilbert space. Liu [6] consider the Mann iteration for strongly pseudocontractive maps defined on a Banach space which is more general than an Hilbert space. This paper generalized the result of Liu by considering the Mann iteration with errors for strongly pseudocontractive maps defined on a Banach space.

2.0 Main result

Let us denote $F(T) = \{p: Tp = p\}$. We now give the following result.

***Theorem* 2.1**

Let X be a Banach space, and let K be a nonempty closed convex and bounded subset of a Banach X. Let $T: K \to K$ be a Lipschitzian strictly pseudo contractive mapping. If $F(T) \neq \phi$, then $\{x_n\} \subset K$ generated by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n + e_n, x_0 \in k,\ n \geq 0$, with $\{\alpha_n\} \subset [0,1]$ satisfying

(i) $\alpha_n \to \infty$ and

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ strongly converges to p Є $F(T)$.

Proof

Following the approach in [6] we have,

$$x_n = x_{n+1} + \alpha_n x_n - \alpha_n Tx_n - e_n$$

$$= (1+\alpha_n)x_{n+1} + \alpha_n (I - T - kI)x_{n+1} - (2-x)\,\alpha_n x_{n+1} + \alpha_n x_n + \alpha_n (Tx_{n+1} - Tx_n) - e_n \quad (2.1)$$

$$= (1+\alpha_n)x_{n+1} + \alpha_n (I - T - kI)x_{n+1} - (2-k)\alpha_n [(1-\alpha_n)x_n + \alpha_n Tx_n + e_n] + \alpha_n x_n + \alpha_n (Tx_{n+1} - Tx_n) - e_n \quad (2.2)$$

$$= (1+\alpha_n)x_{n+1} + \alpha_n (I - T - kI)x_{n+1} - (2-k)\alpha_n x_n + (2-K)\alpha_n^2 (x_n - Tx_n) - (2-k)\alpha_n e_n + \alpha_n x_n + \alpha_n (Tx_{n+1} - Tx_n) - e_n$$

$$= (1+\alpha_n)x_{n+1} + \alpha_n (I - T - kI)x_{n+1} - (1-k)\alpha_n x_n + (2-k)\alpha_n^2 (x_n - Tx_n) + \alpha_n (Tx_{n+1} - Tx_n) - e_n - (2-k)\alpha_n e_n \quad (2.3)$$

By using the inequality (1.4) with $x = x_{n+1}$ and $y = p$

$$\|x_{n+1} - p\| \leq \|x_{n+1} - p + \alpha_n [(I - T - kI)x_{n+1} - (I - T - kI)p] = \|x_{n+1} - p + \alpha_n (I - T - kI)(x_{n+1} - p)\|$$

Thus,

$$0 \leq \alpha_n (I - T - kI)\|x_{n+1} - p\|. \quad (2.4)$$

In view of (2.4), inequality (2.3) can be rewritten as

$$\|x_n - p\| \geq (1+\alpha_n) \|x_{n+1} - p\| - (1-k)\alpha_n \|x_n - p\| - (2-k)\alpha_n^2 \|x_n - Tx_n\| - \alpha_n \|Tx_{n+1} - Tx_n\| - \|e_n\| (1 + (2-k)\alpha_n) \quad (2.5)$$

$$(1+\alpha_n)\|x_{n+1} - p\| \leq \|x_n - p\| + (1-k)\alpha_n \|x_n - p\| + (2-k)\alpha_n^2 \|x_n - Tx_n\| + \alpha_n \|Tx_{n+1} - Tx_n\| + \|e_n\| (1 + (2-k)\alpha_n) \quad (2.6)$$

Since T is Lipschitzian, from (1.5) we have

$$\begin{aligned}\|Tx_{n+1}-Tx_n\| &\le L\|x_{n+1}-x_n\| \\ &= L\|(1-\alpha_n)x_n+\alpha_n Tx_n+e_n-x_n\| \\ &= L\|\alpha_n(Tx_n-x_n)+e_n\| \\ &\le \alpha_n L\|Tx_n-x_n\|+e_nL\end{aligned} \tag{2.7}$$

But,

$$\begin{aligned}\|Tx_n-x_n\| &= \|Tx_n-p+p-x_n\| \\ &= \|(Tx_n-p)+(p-x_n)\| \le \|Tx_n-p\|+\|x_n-p\| \\ &= \|Tx_n-Tp\|+\|x_n-p\| \le L\|x_n-p\|+\|x_n-p\| \\ &= (1+L)\|x_n-p\|\end{aligned} \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6) to get

$$\begin{aligned}(1+\alpha_n)\|x_{n+1}-p\| &\le \|x_n-p\|+(1-k)\alpha_n\|x_n-p\|+(2-k)\alpha_n^2(1+L)\|x_n-p\| \\ &\quad + \alpha_n^2 L\|Tx_n-x_n\|+\alpha_n e_nL+e_n(1+(2-k)\alpha_n) \\ &= \|x_n-p\|+(1-k)\alpha_n\|x_n-p\|+(2-k)\alpha_n^2(1+L)\|x_n-p\| \\ &\quad +\alpha_n^2L(1+L)\|x_n-p\|+\alpha_n e_nL+e_n(1+(2-k)\alpha_n) \\ &= [1+(1-k)\alpha_n]\|x_n-p\|+(2-k)\alpha_n^2(1+L)\|x_n-p\| \\ &\quad +\alpha_n^2L(1+L)\|x_n-p\|+\alpha_n e_nL+e_n(1+(2-k)\alpha_n)\end{aligned} \tag{2.9}$$

$$\begin{aligned}\|x_{n+1}-p\| &\le (1+\alpha_n)^{-1}[1+(1-k)\alpha_n]\|x_n-p\|+(1+\alpha_n)^{-1}(2-k)\alpha_n^2(1+L)\|x_n-p\| \\ &\quad +(1+\alpha_n)^{-1}\alpha_n^2L(1+L)\|x_n-p\|+(1+\alpha_n)^{-1}e_n(1+(2-k)\alpha_n)+(1+\alpha_n)^{-1}\alpha_n e_nL\end{aligned} \tag{2.10}$$

Recall that $(1+\alpha_n)^{-1} \le 1-\alpha_n+\alpha_n^2$ and $(1+\alpha_n)^{-1} \le 1$. Hence

$$\begin{aligned}\|x_{n+1}-p\| &\le [1+(1-k)\alpha_n]\left(1-\alpha_n+\alpha_n^2\right)\|x_n-p\|+(2-k)\alpha_n^2(1+L)\|x_n-p\| \\ &\quad +\alpha_n^2L(1+L)\|x_n-p\|+\alpha_n e_nL+e_n[1+(2-k)\alpha_n]\end{aligned} \tag{2.11}$$

$$\begin{aligned}&= \left[(1+(1-k)\alpha_n)\left(1-\alpha_n+\alpha_n^2\right)+(2-k)\alpha_n^2(1+L)+\alpha_n^2L(1+L)\right]\|x_n-p\| \\ &\quad +[\alpha_nL+(1+(2-k)\alpha_n)]e_n\end{aligned} \tag{2.12}$$

$$\begin{aligned}&= \left[(1+(1-k)\alpha_n)\left(1-\alpha_n+\alpha_n^2\right)+(L+2-k)\alpha_n^2(1+L)\right]\|x_n-p\| \\ &\quad +[\alpha_n(L+2-k)+1]e_n \\ &= \gamma_n\|x_n-p\|+w_ne_n\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}\gamma_n &= \left[(1+(1-k)\alpha_n)\left(1-\alpha_n+\alpha_n^2\right)+\alpha_n^2(1+L)(L+2-k)\right], \\ w_n &= [\alpha_n(L+2-k)+1]\end{aligned}$$

Note that $(1+(1-k)\alpha_n)\left(1-\alpha_n+\alpha_n^2\right) = 1-k\alpha_n + k\alpha_n^2 + (1-k)\alpha_n^3 \leq 1-k\alpha_n + k\alpha_n^2 + (1 - k)$

$\alpha_n^2 = 1-k\alpha_n + \alpha_n^2$.

Therefore

$$\gamma_n \leq 1-k\alpha_n + \alpha_n^2 (1+L)(L+2-k) = 1-k\alpha_n + \alpha_n^2 N \quad (2.14)$$

where

$$N = (1+L)(L+2-k).$$

Since $\{\alpha_n\}$ satisfy (i), there exists an integer N such that

$$\alpha_n N \leq k(1-k) \text{ for all } n \geq N \quad (2.15)$$

implying $\alpha_n^2 N \leq \alpha_n k(1-k)$.

Thus,

$$\|x_{n+1} - p\| \leq \left(1-k^2\alpha_n\right)\|x_n - p\| + [\alpha_n (L+2-k)+1]e_n \quad (2.16)$$

with

$$\lambda_n = k^2\alpha_n, \ a_n = \|x_n - p\| \text{ and } \sigma_n = [\alpha_n (L+2-k)+1]e_n$$

and, it follows from Lemma of Weng [10] that the sequences $\{x_n\}$ strongly converges to the unique fixed point p. ■

3.0 Conclusion

We have been able to show that Mann iteration with errors can be used to approximate the fixed point of a Lipschipzian strictly pseudocontractive mapping defined on a closed, convex subset of a Banach space. We note, however that in the proof of Theorem1 of Liu [6] if $e_n \equiv 0$, then our result is the same as that of Liu. Furthermore, the non-Lipschitzian version of our Theorem with no error term was proved in [2] and [6] under the assumption that X is a uniformly smooth Banach space by using the inequality.

$$\|x+y\|^2 \leq \|x\|^2 + 2 \ \text{Re} \ \langle y, Jx\rangle + \max\{\|x\|, 1\}\|y\| \ \beta(\|y\|)$$

It remains an open problem to show that the Mann iteration with errors can be used to approximate the fixed point of non-Lipschitzian strictly pseudocontractive maps.

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