Journal of the Nigerian Association of Mathematical Physics
***Volume* 11 (November 2007), 51 - 56**

On the equivalence of Picard, Mann and Ishikawa iterations for a class of quasi-contractive operators

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Abstract

We show that the Picard, Mann and the Ishikawa iterations are equivalent when applied to a class of quasi-contractive operators. This result generalises that of Soltuz among others.

2000 AMS Mathematics Classification: 47H10

Keywords: quasi-contractive operators, Picard, Mann and Ishikawa iterations

1. 0 Introduction

The three most common iteration schemes used in approximating the fixed points of an operator defined on a normed space are the Picard, Mann and Ishikawa iteration schemes (see [1]). Let X be a normed space, and C a nonempty convex subset of X. Let T be a self map of C, and let $x_o = u_o = p_o \in C$. The Mann iteration (see [5]) is defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n Tu_n \tag{1.1}$$

The Ishikawa iteration (see [4]) is defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n$ (1.2)

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n \tag{1.3}$$

where $\alpha_n \subset (0, 1)$, $\beta_n \subset [0, 1)$. The Picard iteration is given by

$$p_{n+1} = Tp_n \tag{1.4}$$

Osilike [8] introduced the following contractive definition: Let (X, d) be a metric space, there exists $L \geq 0$, $a \in [0, 1)$ such that for each $x, y \in X$,

$$d(Tx,Ty) \leq Ld(x,Tx) + ad(x,y) \tag{1.5}$$

It should be observed that if $L = 2\delta$, a = δ, we obtain the Zamfirescu operator studied by several authors (e.g. see [1-2], [13],[15]), where

$$\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\} < 1 \tag{1.6}$$

Several authors have studied contractive maps satisfying (0.5). For example see [1], [3] and [8]. The fixed point is unique if it exists [8].

The following conjecture was given in [9]: "if the Mann iteration converges, then so does the Ishikawa iteration". A positive answer was given in a series of papers [9-11] showing the equivalence between the Mann and Ishikawa iterations for strongly and uniformly psudocontractive maps. In [13], Soltuz showed that the convergence of the Picard, the Mann and the Ishikawa iterations are equivalent when dealing with the class of quasi-contractive operators called the Zamfirescu operators [15]. This class of operators are independent of the class of strongly pseudo-contractive operators considered in [9-11]. In this paper, we show that the convergence of the Picard, the Mann and the Ishikawa iterations are equivalent when dealing with a class of operators (1.5) which are more general than those considered considered by Soltuz, i.e. the Zamfirescu operators. First we show the equivalence of Mann and Ishikawa iterations; and then we show the equivalence of Picard and Mann iterations.

***Lemma* 1.1** [14]

Let $(a_n)_n$ be a nonnegative sequence which satisfies the following inequality where

$$\lambda_n \in (0,1), \text{ for al } a_{n+1} \leq (1-\lambda_n)\, a_n + \sigma_n \tag{1.7}$$

l $n \geq 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sigma_n = 0(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

***Theorem* 1.2**

Let X be a normed space, C a nonempty, convex, closed subset of X and T a self map of C satisfying (0.5). Suppose T has a fixed point x^*. Then for $u_o = x_o \in C$, the following are equivalent:

(*i*) the Mann iteration (1.1) converges to x^*;

(*ii*) the Ishikawa iteration (1.2 - 1.3) converges to x^*.

Proof

First we show that (*i*) implies (*ii*). Suppose $\lim_{n\to\infty} u_n = x^*$. Then $0 \leq ||x^* - x_n|| \leq ||u_n - x^*|| + ||x_n - u_n||$.

In order to get $\lim_{n\to\infty} x_n = x^*$, it is sufficient to show that

$$\lim_{n\to\infty} ||x_n - u_n|| = 0 \tag{1.8}$$

In view of (1.1 – 1.3), with $x = u_n = y_n$ in (1.5) we have

$$\begin{aligned} || u_{n+1} - x_{n+1}|| &\leq ||(1-\alpha_n)(u_n - x_n) + \alpha_n (Tu_n - Ty_n)|| \\ &\leq (1-\alpha_n)||u_n - x_n|| + a\,\alpha_n ||u_n - y_n|| + \alpha_n L||u_n - Tu_n||. \end{aligned}$$

If we substitute $x = u_n$, $y = x_n$ in (1.5), we have

$$\begin{aligned} ||u_n - y_n|| &\leq ||(1-\beta_n)(u_n - x_n) + \beta_n (u_n - Tx_n)|| \leq (1-\beta_n)||u_n - x_n|| + \beta_n ||u_n - Tx_n|| \\ &\leq (1-\beta_n)||u_n - x_n|| + \beta_n ||u_n - Tu_n|| + \beta_n ||Tu_n - Tx_n|| \\ &\leq (1-\beta_n)||u_n - x_n|| + a\,\beta_n ||u_n - Tu_n|| + \beta_n ||u_n - x_n|| + \beta_n L||u_n - Tu_n|| \\ &= (1-\beta_n + a\,\beta_n)||u_n - x_n|| + \beta_n ||u_n - Tu_n||(1+L). \end{aligned}$$

Consequently we have

$$\begin{aligned} ||u_{n+1} - x_{n+1}|| &\leq (1-\alpha_n)||u_n - x_n|| + a\,\alpha_n (1-\beta_n + a\,\beta_n)||u_n - x_n|| + a\,\alpha_n \beta_n ||u_n - Tu_n||(1+L) + \alpha_n L||u_n - Tu_n|| \\ &= (1-\alpha_n + a\,\alpha_n - a\,\alpha_n \beta_n + a^2 \alpha_n \beta_n)||u_n - x_n|| + (a\,\alpha_n \beta_n (1+L) + \alpha_n L)||u_n - Tu_n|| \\ &\leq 1-\alpha_n (1 - a(1-\beta_n (1-a)))||u_n - x_n|| + (a\,\alpha_n \beta_n (1+L) + \alpha_n L)||u_n - Tu_n|| \end{aligned}$$

Suppose

$$a_n = ||u_n - x_n||,$$

$$\lambda_n = \alpha_n (1 - a(1 - \beta_n (1 - a))) \subset (0,1) \quad \sigma_n = (a\alpha_n \beta_n (1 + L) + \alpha_n L)\|u_n - Tu_n\|.$$

Since x^* is a fixed point of T, *i.e.* $Tx^*=x^*$, and $\lim_{n\to\infty} \|u_n - x^*\| = 0$, then by (1.5) we have

$$0 \leq \|u_n - Tu_n\| \leq \|u_n - x^*\| + \|Tx^* - Tu_n\|$$
$$\leq (a + 1)\|u_n - x^*\| \to 0 \text{ as } n\to\infty.$$

Hence

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0; \text{ that is, } \sigma_n = 0(\lambda_n).$$

By an application of Lemma 1, we have $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Conversely, we show that (*ii*) implies (*i*). Assume $x_n \to x^*$. If $x = y_n$ and $y = u_n$ in (1.5), we have,

$$\|x_{n+1} - u_{n+1}\| \leq \|(1 - \alpha_n)(x_n - u_n) + \alpha_n (Ty_n - Tu_n)\|$$
$$\leq (1 - \alpha_n)\|x_n - u_n\| + \alpha_n \|Ty_n - Tu_n\|$$
$$\leq (1 - \alpha_n)\|x_n - u_n\| + a\alpha_n \|y_n - u_n\| + \alpha_n L\|y_n - Ty_n\|.$$

Also,

$$\|y_n - u_n\| \leq \|(1 - \beta_n)(x_n - u_n) + \beta_n (Tx_n - u_n)\|$$
$$\leq (1 - \beta_n)\|x_n - u_n\| + \beta_n \|Tx_n - u_n\|$$
$$\leq (1 - \beta_n)\|x_n - u_n\| + \beta_n \|Tx_n - x_n\| + \beta_n \|x_n - u_n\|$$
$$\leq \|x - u_n\| + \beta_n \|Tx_n - x_n\|.$$

Hence

$$\|x_{n+1} - u_{n+1}\| \leq (1 - \alpha_n)\|x_n - u_n\| + a\alpha_n (\|x_n - u_n\| + \beta_n \|Tx_n - x_n\|) + \alpha_n L\|y_n - Ty_n\|$$
$$\leq (1 - \alpha_n + a\alpha_n)\|x_n - u_n\| + a\alpha_n \beta_n \|Tx_n - x_n\| + \alpha_n L\|y_n - Ty_n\|$$

Suppose

$$a_n = \|x_n - u_n\|,\ 1 - \lambda_n = (1 - \alpha_n + a\alpha_n) \subset (0, 1);$$
$$\sigma_n = a\alpha_n \beta_n \|Tx_n - x_n\| + \alpha_n L\|y_n - Ty_n\|$$

Since x^* is a fixed point of T, *i.e.* $Tx^* = x^*$ and $\lim_{n\to\infty} \|x_n - x^*\| = 0$, then by (1.5) we have

$$0 \leq \|x_n - Tx_n\| \leq \|x_n - x^*\| + \|Tx^* - Tx_n\| \leq (a + 1)\|x_n - x^*\| \to 0 \text{ and } n \to \infty$$

Also

$$0 \leq \|y_n - Ty_n\| \leq \|y_n - x^*\| + \|Tx^* - Ty_n\| \leq (a + 1)\|y_n - x^*\|$$
$$\leq (a+1)\{(1 - \beta_n)\|x_n - x^*\| + \beta_n \|Tx_n - Tx^*\|\}$$
$$\leq (a + 1)(1 - \beta_n (1 - a)\|x_n - x^*\| \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - Ty_n\| = 0, \text{ that is, } \sigma_n = 0(\lambda_n).$$

By an application of Lemma 1, we have that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Thus we get $\|x^* - u_n\| \leq \|x_n - u_n\| + \|x_n - x^*\| \to 0$ as $n \to \infty$. ■

Theorem 1.3

Let X normed, C a nonempty, convex,closed subset of X and T a self map of C satisfying (1.5). Let $u_o = p_o \in C$. Suppose T has a fixed point x^*. Then, the following are equivalent:

(*i*) the Mann iteration (1.1) converges to x^* implies that the Picard iteration (1.4) converges to x^*;

(*ii*) the Picard iteration (1.4) converges to x^* and $\lim_{n\to\infty} \|p_{n+1} - p_n\| = 0$

implies the Mann iteration converges to x^*.

Proof

Assume that the Mann iteration converges, *i. e.* $u_n \to 0$. We are to prove that the Picard iteration converges too. If we put $x = u_n$ and $y = p_n$ in (1.5), in view of (1.1) and (1.4), we have

$$\begin{aligned}
||u_{n+1} - p_{n+1}|| &\le (1 - \alpha_n)||u_n - Tp_n|| + \alpha_n ||Tu_n - Tp_n|| \\
&\le (1 - \alpha_n)||u_n - Tp_n|| + a\alpha_n ||u_n - p_n|| + \alpha_n L||Tu_n - u_n|| \\
&= (1 - \alpha_n)||u_n - p_{n+1}|| + a\alpha_n ||u_n - p_n|| + \alpha_n L||Tu_n - u_n|| \\
&\le (1 - \alpha_n)||u_{n+1} - p_{n+1}|| + (1 - \alpha_n)||u_{n+1} - u_n|| + \alpha_n a||u_n - p_n|| + \alpha_n L||Tu_n - u_n||
\end{aligned}$$

Hence,

$$\alpha_n || u_{n+1} - p_{n+1}|| \le (1-\alpha_n)|| u_{n+1} - u_n|| + \alpha_n a||u_n - p_n|| + \alpha_n L ||Tu_n - u_n||$$

Therefore,

$$||u_{n+1} - p_{n+1}|| \le a||u_n - p_n|| + L||Tu_n - u_n|| + \frac{1-\alpha_n}{\alpha_n}||u_{n+1} - u_n|| \tag{1.9}$$

In view of (1.1),

$$\frac{1-\alpha_n}{\alpha_n}||u_{n+1} - u_n|| = (1 - \alpha_n)||Tu_n - u_n|| \tag{1.10}$$

Thus (1.9) becomes

$$||u_{n+1} - p_{n+1}|| \le a||u_n - p_n|| + (L+1 - \alpha_n)||Tu_n - u_n|| \tag{1.11}$$

Suppose $a_n = ||u_n - p_n||$; $1-\lambda_n = a \in (1,1)$; $\sigma_n = (L+1-\alpha_n)||Tu_n - u_n||$. An application of Lemma 1 shows that

$$\lim_{n\to\infty} ||u_n - p_n|| = 0. \text{ Thus } ||x^* - p_n|| \le ||u_n - p_n|| + ||u_n - x^*|| \to 0 \text{ as } n \to \infty.$$

Conversely assume that the Picard iteration converges, we want to show that the Mann iteration converges to

$$||u_n - Tp_n|| \le ||u_n - p_n|| + ||p_n - Tp_n||.$$

In view of (1.11) and (1.4), if we set $x = p_n$, $y = u_n$ in (1.5) we get

$$\begin{aligned}
||u_{n+1} - p_{n+1}|| &\le (1 - \alpha_n)||u_n - Tp_n|| + \alpha_n ||Tp_n - Tu_n|| \\
&\le (1-\alpha_n)||u_n - p_n|| + (1-\alpha_n)||p_n - Tp_n|| + \alpha_n a||p_n - u_n|| + \alpha_n L||Tp_n - p_n|| \\
&= (1-\alpha_n + \alpha_n a)||p_n - u_n|| + (1+\alpha_n L - \alpha_n)||p_n - Tp_n|| \\
&= (1-(1-a)\alpha_n)||u_n - p_n|| + (1+\alpha_n L - \alpha_n)||p_n - Tp_n||
\end{aligned}$$

Suppose, $a_n = ||u_n - p_n||$; $\lambda_n = (1 - a)\alpha_n \in (0,1)$; $\sigma_n = (1+\alpha_n L - \alpha_n)||p_n - p_{n+1}||$.

An application of Lemma 1 shows that $\lim_{n\to\infty} ||u_n - p_n|| = 0$. Thus $||x^* - u_n|| \le ||u_n - p_n|| + ||p_n - x^*|| \to 0$ as $n \to \infty$. A combination of Theorem 1.2 and Theorem 1.3 yields the following result.

■

***Corollary* 1.3**

Let X be a normed space, C a nonempty, convex, closed subset of X and T a self map of C satisfying (1.5). Suppose T has a fixed point denoted by x^*. Then for $u_o = x_o = p_o \in C$, the following are equivalent:

(*i*) the Mann iteration (1.1) converges to x^*;
(*ii*) the Ishikawa iteration (1.2 - 1.3) converges to x^*;
(iii) the Picard iteration (1.4) converges to x^*.

***Corollary* 1.4 [13].**

Let X be a normed space, C a nonempty, convex, closed subset of X and T a self map of C such that T is a Zamfirescu operator. Suppose T has a fixed point denoted by x^*. Then for $u_o = x_o = p_o \in C$, the following are equivalent:

(i) the Mann iteration (1.1) converges to x^*;

(ii) the Ishikawa iteration (1.2 - 1.3) converges to x^*;

(iii) the Picard iteration (1.4) converges to x^*.

2.0 Conclusion

1. The technique of the proof of our results is due to Soltuz [13] which was partly due to Berinde [2]. Theorem 2, apart from being a generalisation of [13, Theorem 2] is stronger than that of Soltuz since the assumption that $\lim_{n\to\infty} \frac{\| u_{n+1} - p_{n+1} \|}{\alpha_n} = 0$ is no more necessary.

2. Rhoades et al [12] proved the equivalence of Mann and Ishikawa iterations for *generalised contractions* . But the operator satisfying (0.5) need not be a generalised contraction. For example, the operator satisfying (0.5) need not have a fixed point [8] while the generalised contraction studied in [12] always have a unique fixed point. It should also be noted that Rhoades proved only the equivalence of Mann and Ishikawa iterations while our own included that of the Picard iteration.

3. The implication of this result is that when dealing with a class of mappings that satisfies (0.5) such that the fixed points exist (e.g. the Zamfirescu's operators), since the Picard iteration converges exponentially to the fixed point of T, and thus faster than the Mann and Ishikawa iterations which converges linearly, (e.g. see [2], [6-7]), it is no more necessary to use the Mann or the Ishikawa iterations to approximate the fixed point of T. Rather we use the simpler and faster Picard iteration.

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