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## Subgroups of Group of homotopy spheres

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***Abstract***

***Let $\theta^n$ denote the group of h-cobordism classes of homotopy n-sphere under the connected sum operation. H(p, q) is the subgroup of $\theta^p$ consisting of those homotopy p-spheres $\sum^p$ such that $\sum^p \times S^q$ is diffeomorphic to $S^p \times S^q$. Also $bP_{p+1}$ is the subgroup of homotopy p-sphere which bounds parallelizable manifolds. In this paper, we will prove that $\frac{H(p,q)}{bP_{p+1}}$ is isomorphic to the Cokernel of Hopf-Whitehead homorphism $J : \Pi_p S0(q+1) \to \Pi_{p+q+1}(S^{q+1})$.***

## 1.0 Introduction

The notion of homotopy sphere goes back to John Milnor [9] when he constructed a 7-sphere which is homeomorphic to the standard 7-sphere but not diffeomorphic to it. There after some of the finest mathematicians contributed to this area of differential topology. In 1960, Michel Kervaire and John Milnor came out with their famous paper "Groups of homotopy Spheres I" [7] where they showed that the collection of homotopy *n*-sphere form Abelian group denoted by $\theta^n$. Operation on the set is connected sum # which they defined. They also calculated the order of $\theta^n$ for some integer *n*. R. Desapio [3], Morris. W Hirsh [4], W.C. Hsiang, J, Levine and R.H. Sczarba [5], J. Munkres [10], S. Smale [13]. E.C. Zeeman [15]. K. Kawakubo [6], Hajime Sato [11], Edward .C. turner [14], A. Kosinski [2], and Reinhard Schultz [12] are among the 20th century finest mathematicians who contributed to the development of differential topology and in particular the study and application of homotopy sphere in the classification of different types of smooth manifolds.

In [1] we show that $bP_{p+1}$ is a subgroup of $\phi_p^2$. In this paper we will show that the subgroups $H(p, q)$ and $\phi_p^{q+1}$ of $\theta^p$ are actually equal i.e. $H(p, q) = \phi_p^{q+1}$. It will then follow that $bP_{p+1}$ is a subgroup of $H(p, q)$. Since $\theta^p$ is an abelian group then it follows that every subgroup of $\theta^p$ is a normal subgroup. Thus we will show that the quotient group $\frac{H(p,q)}{bP_{p+1}}$ is a isomorphic to the Cokernel of Hopf-Whitehead homorphism

$J : \Pi_p S0(q+1) \to \Pi_{p+q+1}(S^{q+1})$ provided $2q > p - 1$. This is of interest because the groups $bP_{p+1}$ have been determined in many cases in [7]. From our result here it now becomes easy to determined the image of the Hopf-Whitehead homorphism since we can determine the group $\frac{H(p,q)}{bP_{p+1}}$.

## 1.0 Preliminaries

$S^n$ denotes the unit $n$-sphere with the usual smooth structure in the Euclidean $(n + 1)$ – space $R^{n+1}$, # means connected sum along boundary as defined by Milnor and Kervaire [8]. If $M_1$ and $M_2$ are two connected $n$ – manifolds, we remove the interior of the unit disk $D^n$ from each of the manifolds $M_1$ and $M_2$. We now join the two manifolds along their common boundary to get $M_1 \# M_2$. A homotopy $n$-sphere is a closed $n$ – manifold which has the homotopy type of the standard $n$-sphere $S^n$. We denote an homotopy $n$-sphere $\sum^n$.

## 2 0 Groups and subgroups

Let $\theta^n$ denote the $h$-cobordism class of homotopy $n$-sphere. In [8] John Milnor and Michel Kervaire proved that $\theta^n$ is an abelian group with connected sum # as the operation.

***Definition* 2.1**

*Let H(p, q) denote the subset of $\theta^p$ consisting of those homotopy p-spheres $\sum^p$ such that $\sum^p$ x $S^q$ is diffeomorphic to $S^p$ x $S^q$. Let $\Phi_p^{q+1}$ denote the homotopy p-spheres $\sum^p$ which embed in the Euclidian (p + q + 1)–space $R^{p+q+1}$ with trivial normal bundle.*

***Lemma* 2.2**

$H(p, q)$ is a subgroup of $\theta^p$

***Proof***

Let $\sum_1^p$ then $\sum_1^p$ x $S^q$ is diffeomorphic to $S^p$ x $S^q$, $\sum_1^p$ embeds in $(p + q + 1)$ – Disc with trivial normal bundle, it follows that $\sum_1^p$ embeds in the interior of the $(p + q + 1)$ – disc in sss with trivial normal bundle. Obviously $\sum_1^p$ embeds in $\sum_2^p$ x $D^{q+1}$ with trivial normal bundle and hence we can form the connected sum $\sum_1^p \# \sum_2^p$ in $\sum_2^p$ x $D^{q+1}$ so that $\sum_1^p \# \sum_2^p$ has a trivial normal bundle in $\sum_2^p$ x $D^{q+1}$. Notice that $\sum_1^p \# \sum_2 \to \sum_2^p$ x $D^{q+1}$ is an homotopy equivalence, it then follows by Smale [13] that $(\sum_1^p \# \sum_2^p)$ x $D^{q+1}$ is diffeomorphic to $\sum_2^p$ x $D^{q+1}$. Thus $(\sum_1^p \# \sum_2^p)$ x $S^q$ is diffeomorphic to $\sum_2^p$ x $S^q$. From this result, it follows that if $\sum_1^p$, $\sum_2^p \in H(p, q)$ then $\sum_1^p \# \sum_2^p \in H(p, q)$.Obviously $S^p \in H(p,q)$. Thus $H(p, q)$ us a subgroup of $\theta^p$. ■

***Lemma* 2.3**

If $q \geq 2$, then $H(p, q) = \Phi_p^{q+1}$

***Proof***

If $\sum^p \in H(p,q)$ then $\sum^p$ x $S^q$ is diffeomorphic to $S^p$ x $S^q \subset R^{p+q+1}$ hence it follows that $\sum^p$ embeds in $R^{p+q+1}$ with trivial normal bundle, hence $\sum^p \in \Phi_p^{q+1}$. Thus $H(p,$

$q) \subset \Phi_p^{q+1}$. Conversely, $\sum^p \in \Phi_p^{q+1}$ then $\sum^p$ embeds in $R^{p+q+1}$ with trivial normal bundle. By Levine [7], the embedding is isotopic to an embedding $\sum^p \to S^p \times D^{q+1}$. Since these two manifolds are homotopically equivalent then it follows by Smale [13] that $\sum^p \times D^{q+1}$ is diffeomorphic to $S^p \times D^{q+1}$. Hence $\sum^p \times S^q$ is diffeomorphic to $S^p \times S^q$. This shows that $\sum^p \in H(p,q)$ and so $\Phi_p^{q+1} \subset H(p, q)$. Thus $H(p, q) = \Phi_p^{q+1}$. ■

***Definition* 2.4**

*Let $bP_{p+1}$ denote set of all homotopy n – spheres which bound parallelizable manifolds. In* [8] *Milnor and Kervaire showed that $bP_{p+1}$ is a subgroup of $\theta^n$. This subgroup of $\theta^n$ is of interest since its values have largely been determin*ed i*n* [8]. T*his subgroup is not always trivial subgroup although if n is even $bP_{p+1}$ is trivial.*

In [1], we showed that $bP_{p+1} \subset \Phi_p^2$ and since $H(p, q) = \Phi_p^{q+1}$, it follows that $bP_{p+1} \subset H(p, q)$ $q \geq 2$. We now wish to determine $\frac{H(p,q)}{bP_{p+1}}$ in terms of Cokernel of Hopf-Whitehead homomorphism $J = J : \Pi_p S0(q+1) \to \Pi_{p+q+1}(S^{q+1})$.

***Theorem* 2.5**

$\frac{H(p,q)}{bP_{p+1}}$ *is isomorphic to COK* $\left(J_p^q\right)$

***Proof***

Since $H(p, q) = \Phi_p^{q+1}$ for $q \geq 2$ it follows from [5] that the following sequence is exact $0 \to bP_{p+1} \xrightarrow{\vartheta} H(p,q) \xrightarrow{\psi} COK\left(J_p^q\right) \to 0$. Since the sequences is short exact then $\vartheta$ is one-to-one and $\psi$ is onto. It then follows that $im(\vartheta)$ is isomorphic to $bP_{p+1}$. Since the sequence is exact $im(\vartheta) = Ker(\psi) = bP_{p+1}$. From the short exact sequence $\frac{H(p,q)}{\ker(\Psi)} = im(\Psi)$.. But $im(\vartheta) = Cok(J_p^q)$. ■

Since $\psi$ is onto hence $\frac{H(p,q)}{bP_{p+1}} = Cok(J_p^q)$.

## 3.0 Conclusion

From the theorem above it is now possible to determine the image of Hopf-Whitehead homorphism $J : \Pi_p S0(q+1) \to \Pi_{p+q+1}(S^{q+1})$ using the value $\frac{H(p,q)}{bP_{p+1}}$ already known. For example $\frac{H(8,4)}{bP_9} = Z_2$ and it then follows that image $(J_9) = cok(J_9) = Z_2$. Also since order of $H(11,3) = |H(11,3)| = 992$ and order of $bP_{12} = |bP_{12}| = 992$ then it follows that $\left|\frac{H(11,3)}{bP_{12}}\right| = |J_{11}| = \frac{992}{992} = 1$. It follows that $J_{11}$ is a trivial homomorphism.

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