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Diffeomorphism groups of connected sum of three products of spheres

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Abstract

In this paper, we will prove that the order of the group $\tilde{\pi}_0(M^+_{3(p,q)})$ ***is 3 times the order of the group*** $\pi_p SO(q+1) \oplus \pi_q SO(p+1) \oplus \theta^{p+q+1}$ ***where*** $p < q$ ***and*** $M^+_{3(p,q)} \approx Diff^+ (S^p \times S^q \# S^p \times S^q \# S^p \times S^q)$. $Diff^+(M)$ ***is the diffeomorrphism of*** M ***onto itself which induces identity homomorphism on homology.***

1.0. Introduction

Let M be an oriented smooth manifold and let $Diff(M)$ denote the group of all orientation preserving diffeomorphisms of M. We say that two elements f_0, $f_1 \in Diff(M)$ are pseudo-diffeotopic if there exists $F \in Diff(M \times I)$ such that $F(x,0) = (f_0(x),0)$ and $F(x,1) = (f_1(x),1)$, where $x \in M$.

The pseudo-diffeotopy classes of diffeomorphism on M form a group to be denoted by $\tilde{\pi}_0(Diff(M))$. Since pseudo-diffeotopic diffeomorphisms induce equal automorphism on homology then we have a well defined homomorphism, $\Phi : \tilde{\pi}_0(Diff(M)) \rightarrow Auto(H_*(M))$ where $Auto(H_*(M))$ denotes the group of dimension-preserving automorphisms of $H_*(M)$. We will denote by $Diff^+(M)$ the subgroup of $Diff(M)$ consisting of all diffeomorphism $f \in Diff(M)$ which induce identity map on all homology groups. It is easily seen that kernel of Φ denoted by $\ker(\Phi) = Diff^+(M)$.

For convenience, we will use the following notation.

$$M_{3(p,q)} = Diff(S^p \times S^q \# S^p \times S^q \# S^p \times S^q)$$

$$M_{2(p,q)} = Diff(S^p \times S^p \# S^p \times S^q)$$

$$M_{(p,q)} = Diff(S^p \times S^p)$$

We denote by $\tilde{\pi}_0(M^+_{3(p,q)})$ the pseudo-diffeotopy class of diffeomorphism in $M_{3(p,q)}$ which induce identity automorphism on homology.

In [5], Hajime Sato investigated $\tilde{\pi}_0(M^+_{(p,q)})$ and later applied it in the classification of certain classes of manifolds. In [1], this author investigated $M^+_{2(p,q)}$, thereby extending Hajime Sato's result to diffeomorphism of connected sum of two product of spheres.

In this paper, we will investigate the structure of the group $\tilde{\pi}_0(M^+_{3(p,q)})$, $p<q$. This result will be useful in the classification of some smooth manifolds. θ^n denotes the group of h-cobordism classes of homotopy n-sphere under the connected sum operation. $\#$ denotes connected sum along boundary as defined by J. Milnor and M. Kervaire [3].

2.0. Preliminaries

In this section, we will discuss some of known results and terminologies which will be needed to prove the theorems in the rest of the paper.

Let M^n be an n-smooth manifold. Suppose we have an embedding $f: S^p \times D^{q+1} \to M$, where $p+q+1=n$, and let $M_0 = M - \text{int}(f(S^p \times D^{q+1}))$, i.e., we remove the interior of image $S^p \times D^{q+1}$ under the map f, we then replace it by $D^{p+1} \times S^q$ to have $M' = M_0 \bigcup_f D^{p+1} \times S^q$ by identifying $(x,y) \in S^p \times S^q \subset D^{p+1} \times S^q$ with $f(x,y) \in \partial f \ (S^p \times D^{q+1}) \subset M$. We will say that we obtain M' from M by spherical modification of the form $[p+1, q+1]$. It is easily seen that by reversing the process, we can obtain M from M' by performing spherical modification of the form $[q+1, p+1]$. This procedure is due to J. Milnor [2].

***Lemma* 1.1**

S^{p+q+r} is diffeomorphic to $D^{p+q+1} \times S^{r-1} \bigcup_{id} S^{p+q} \times D^r$, *where $2 \le p \le q \le r$ and id = identity map: $S^{p+q} \times S^{r-1} \to S^{p+q} \times S^{r-1}$*.

Proof

By [3] Milnor and Kervaire showed that a simply connected manifold is h-cobodant to a sphere if and only if it bounds a contractible manifold. Boundary of $(D^{p+q+1} \times D^r) = \delta(D^{p+q+1} \times D^r) = D^{p+q+1} \times S^{r-1} \cup S^{p+q} \times D^r$. However, these two copies of the boundary of $D^{p+q+1} \times D^r$ have $S^{p+q} \times S^{r-1}$ in common. Hence, $\partial(D^{p+q+1} \times D^r) = D^{p+q+1} \times S^{r-1} \bigcup_{id} S^{p+q} \times D^r$. It follows that $D^{p+q+1} \times S^{r-1} \bigcup_{id} S^{p+q} \times D^r$ is h-cobordant to S^{p+q+r} since it bounds the contractible manifold $D^{p+q+1} \times D^r$. Since $p+q+r \ge 6$ by [5], S^{p+q+r} is diffeomorphic to $D^{p+q+1} \times S^{r-1} \bigcup_{id} S^{p+q} \times D^r$. ■

3.0 The Group $\pi_0(M^+_{3\,(p,q)})$

In [4], Hajime Sato showed that the order of the group $\pi_0(M^+_{(p,q)})$ equals the order of the group $\pi_p SO(q+1) \oplus \pi_q SO(p+1) \oplus \theta^{p+q+1}$, where $p<q$. In [1], this author showed that the order of the group $\pi_0(M^+_{2(p,q)})$ is twice the order of $\pi_p SO(q+1) \oplus \pi_q SO(p+1) \oplus$ θ^{p+q+1}. To investigate $M_{3(p,q)} = \pi_0 Diff^+(S^p \times S^q \# S^p \times S^q \# S^p \times S^q)$, we define a homo-morphism $g: \pi_0(M^+_{3(p,q)}) \to \pi_p SO(q+1)$ defined as follows. Let $[f] \in \pi_0(M^+_{3(p,q)})$, then $\Phi(f)$ = identity. $i(S^p \times \{x_0\})$ is the usual embedding of $S^p \times \{x_0\}$ into = $S^p \times S^q \# S^p \times S^q$ $S^p \times S^q$, where x_0 is a fixed point in S^q which is far removed from the connected sum. It follows that $i(S^p \times \{x_0\})$ represents a generator of the homology $= \pi_0 Diff^+(S^p \times S^q \# S^p \times$ $S^q \# S^p \times S^q) = Z \oplus Z \oplus Z$. Since $[f] \in \pi_0(M^+_{3(p,q)})$, then $\Phi(f)$ is identity automorphism on $H_*(S^p \times S^q \# S^p \times S^q \# S^p \times S^q)$. It follows that $f(S^p \times \{x_0\})$ is homologous to $i(S^p \times \{x_0\})$. Since $p<q$ and by Hurewicz theorem, f and i are homotopic and in fact are diffeotopic. By the tubular neighbourhood theorem, f is diffeotopic to a map f'' such that $f''(S^p \times D^q) = S^p \times D^q$, where $f''(x,y) = (x, \alpha(f'')(x)\cdot y)$ and $\alpha(f''): S^p \to SO(q)$. If $i: SO(q) \to SO(q+1)$ is the inclusion map and $i_*: \pi_p SO(q) \to \pi_p SO(q+1)$ is the induced homomorphism on the homotopy groups. We define $g[f] = i_* \alpha(f'')$.

Lemma 2.1

g is well-defined.

Proof

Let $h \in [f] \in \pi_0(M^+_{3(p,q)})$, then f and h are pseudo-diffeotopic in $M^+_{3(p,q)}$ and so $f^{-1}h \in M^+_{3(p,q)}$ is pseudo-diffeotopic to the identity. Let $g(f) = i_*\alpha(f')$ and $g(h) = i_*\alpha(h')$, then we have $f(x,y) = (x, \alpha(f')x\cdot y)$ and $h(x,y) = (x, \alpha(h')x\cdot y)$, where in each $(x,y) \in S^p \times D^q$ $\subset (S^p \times S^q)$ and $h(x,y) = (x,y)$ and $f(x,y) = (x,y)$ on $(S^p \times S^q)_2$ and $(S^p \times S^q)_3$. Thus, for (x,y) Є $S^p \times D^q$, $f^{-1}h(x,y) = (x, \alpha(f')^{-1}\alpha(h')x\cdot y)$. We define

$$f^{-1}h(x,y) = \begin{cases} (x, i_*\alpha(f')^{-1} \cdot i_*(h')(x)\cdot y) \text{ if } (x,y) \in (S^p \times S^q)_1 \\ (x,y) \text{ if } (x,y) \in (S^p \times S^q)_2 \\ (x,y) \text{ if } (x,y) \in (S^p \times S^q)_3 \end{cases}$$

where the subscripts 1, 2 and 3 denote the first, second and third summands of $S^p \times S^q \# S^p \times S^q \# S^p \times S^q$

Since $f^{-1}h$ is pseudo-diffeotopic to the identity, it follows that $i{*}\alpha(f')^{-1} \cdot i{*}(h')$ is identity. Hence, $i{*}\alpha(f') = i{*}\alpha(h')$. Thus, g is well-defined. ■

***Lemma* 2.2**

g is an homomorphism.

Proof

Let $[f]$, $[h] \in \tilde{\pi}_0(M^+_{3(p,q)})$, then $g[f] = i{*}\alpha(f')$ and $g[h] = i{*}\alpha(h')$. Since $i{*}$ is a homomorphism, then

$$g([f]\cdot[h]) = g(f \cdot h) = i\alpha(f' \cdot h') = i{*}\alpha(f') \cdot i{*}\alpha(h') = g[f] \cdot g[h'].$$

Therefore, g is an homorphism. ■

***Lemma* 2.3**

g is surjective.

Proof

To show that g is surjective is to show that $g(\pi_0(M^+_{3(p,q)}) = i{*}(\pi_p SO(q))$. From the definition of g, it is clear that $g(\tilde{\pi}_0(M^+_{3(p,q)}) \subset i{*}\pi_p SO(q)$, we only need to show that $i{*}(\pi_p SO(q)) \subset g(\tilde{\pi}_0(M^+_{3(p,q)})$. Let $\alpha \in i{*}\pi_p SO(q)$, where $[a] = \alpha$ and $a : S^p \to SO(q+1)$, we define

$$f(x,y) = \begin{cases} (x, a(x) \cdot y \text{ if } (x,y) \in (S^p \times S^q)_1 \\ (x,y) \text{ if } (x,y) \in (S^p \times S^q)_2 \\ (x,y) \text{ if } (x,y) \in (S^p \times S^q)_3 \end{cases}$$

Since $a \in i{*}(\pi_p SO(q))$, then f induces identity on homology and so $f \in \pi_0(M^+_{3(p,q)})$.

Thus, $i{*}(\pi_p SO(q)) \subset \pi_0(M^+_{3(p,q)})$ and so $g(\pi_0(M^+_{3(p,q)}) = i{*}(\pi_p SO(q))$. In fact, from the dimension restriction, i.e., $p < q$, it follows that $\pi_p(S^q) = 0$. We consider the exact sequence $\cdots \to \pi_{p+1}(S^q) \to \pi_p SO(q) \to \pi_p SO(q+1) \to \pi_p(S^q) \to \cdots$ Since $\pi_p(S^q) = 0$, it follows that i_* is an epimorphism. Hence, $i{*}(\pi_p SO(q)) = \pi_p SO(q+1)$. In fact, if $p < q-1$, then $\pi_{p+1}(S^q) = 0$ and in this case, i_* is an isomorphism and so g is surjective. ■

***Lemma* 2.4**

If $u \in \ker(g)$, then there exists $f \in M^+_{3(p,q)}$ such that $f \in [u]$ and f is identity on $S^p \times D^q$.

Proof

From Lemma 2.3, if $p < q-1$, then $\pi_{p+1}(S^q) = \pi_p(S^q) = 0$. Hence, i_* is an isomorphism. Since $f \in \ker(g)$, it then follows that $g(u) = i_*\alpha(f') = 0$. Since i_* is an isomorphism then, $\alpha(f') = 0$, where $\alpha(f') \in \pi_p SO(q)$. Hence, f is identity on $S^p \times D^q$.

We need to compute $\ker(g)$. To do this, we define a homomorphism

$$l : \ker(g) \to \pi_0(M^+_{2(p,q)}) = \pi_0(Diff^+(S^p \times S^q \# S^p \times S^q))$$

Let $f \in \ker(g)$. Then, by Lemma 3.4, f is identity on $S^p \times D^q$. We therefore have a map $f : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \# (S^p \times S^q)_3 \to (S^p \times S^q)_4 \# (S^p \times S^q)_5 \# (S^p \times S^q)_6$ f is identity on $S^p \times D^q \subset (S^p \times S^q)_1$ using spherical modification technique introduced by Milnor in [2] and [3] on the domain $(S^p \times S^q)_1 \# (S^p \times S^q)_2 \# (S^p \times S^q)_3$ which removes the interior of $S^p \times D^q \subset (S^p \times S^q)_1$ and replaces it with $D^{p+1} \times S^{q-1}$. By Lemma 1.1, $S^p \times D^q \bigcup_{id} D^{p+1} \times S^{q-1}$ is diffeomorphic to S^{p+q}. Thus, the domain of f becomes $S^{p+q} \#$ $(S^p \times S^q)_2 \# (S^p \times S^q)_3 = (S^p \times S^q)_2 \# (S^p \times S^q)_3$.

For the image, since f is the identity on $S^p \times D^q$, we can assume that $f(S^p \times D^q) =$ $S^p \times D^q \subset (S^p \times S^q)_4$. We similarly perform spherical modification on the range $(S^p \times S^q)_4$ $\# (S^p \times S^q)_5 \# (S^p \times S^q)_6$ by removing the interior of $S^p \times D^q$ from $(S^p \times S^q)_4$ and replace it with $D^{p+1} \times S^{q-1}$ to have $S^p \times D^q \bigcup_{id} D^{p+1} \times S^{q-1}$, which by Lemma 1.1 equals S^{p+q}. Thus, the range becomes

$$S^{p+q} \# (S^p \times S^q)_5 \# (S^p \times S^q)_6 = (S^p \times S^q)_5 \# (S^p \times S^q)_6 .$$

It therefore follows that after the spherical modification we are left with a diffeomorphism $f' : S^p \times S^q \# S^p \times S^q \to S^p \times S^q \# S^p \times S^q$. Clearly, $f' \in M^+_{2(p,q)}$ and so we define $l[f] = [f']$. ■

***Lemma* 2.5**

l is surjective.

Proof

Let $h' \in M^{+}_{2(p,q)}$. Then, we will find $h \in M^{+}_{3(p,q)}$ such that $i(h) = h'$. We define

$$h(x,y) = \begin{cases} (x,y) \text{ if } (x,y) \in (S^p \times S^q)_1 \\ h'(x,y) \text{ if } (x,y) \in (S^p \times S^q)_2 \# (S^p \times S^q)_3 \end{cases}$$

Since h is identity on $(S^p \times S^q)_1$, then h is identity on $S^p \times D^q \subset (S^p \times S^q)_1$. Since $h' \in M^{+}_{2(p,q)}$, then $h \in M^{+}_{3(p,q)}$ and so $h \in \ker(g)$ and clearly, $i(h) = h'$. Thus, l is surjective. ■

Recall from [1] the homomorphism $N : \ker(g) \to \pi_0 Diff^{+}(S^p \times S^q)$ and in [4], Sato defined an homomorphism $B : \pi_0 Diff^{+}(S^p \times S^q) \to \pi_p SO(q+1)$. In [1], we showed that $\ker N$ is in one-to-one correspondence with $\ker B$ and Sato gave a computation of $\ker B$. Here we will show the following.

***Lemma* 2.6**

$\ker(l)$ i*s in one-to-one correspondence with* $\ker(N)$.

Proof

Let $f \in \ker(L)$. Then, $L(f) = f'$ is identity in $M^{+}_{2(p,q)}$. Since $f \in \ker(g)$, it follows by Lemma 2.4 that f is identity on $S^p \times D^q$ $f : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \# (S^p \times S^q)_3 \to (S^p \times S^q)_4 \# (S^p \times S^q)_5 \# (S^p \times S^q)_6$ since $f(S^p \times D^q) = S^p \times D^q$. We then perform spherical modification on $(S^p \times S^q)_1$ to remove the interior of $S^p \times D^q$ from $(S^p \times S^q)_1$ and attach $D^{p+1} \times S^{q-1}$ to have $S^p \times D^q \underset{id}{\cup} D^{p+1} \times S^{q-1} = S^{p+q}$ by Lemma 1.1. We similarly perform spherical modification on $(S^p \times S^q)_4$ by removing the interior of $S^p \times D^q$ from $(S^p \times S^q)_4$ and attaching $D^{p+1} \times S^{q-1}$ on their common boundary to have $S^p \times D^q \underset{id}{\cup} D^{p+1}$
$\times S^{q-1} = S^{p+q}$. We now have a map

$$f'' : (S^p \times S^q)_2 \# (S^p \times S^q)_3 \to (S^p \times S^q)_5 \# (S^p \times S^q)_6$$

which is identity on each summand. Hence, $f' \in \ker N$. Conversely, let $f \in \ker(N)$. Then $f \in M^{+}_{2(p,q)}$. Then f induces identity on $S^p \times S^q$. We define a diffeomorphism

$$f'(x,y) = \begin{cases} (x,y) \text{ if } (x,y) \in (S^p \times S^q) \\ f(x,y) \text{ if } (x,y) \in (S^p \times S^q)_2 \# (S^p \times S^q)_3 \end{cases}$$

Clearly, $f' \in \ker(g)$ and since it induces identity on $M^{+}_{2(p,q)}$, then $f' \in \ker l$.

By Lemma 2.4, since l is surjective, we have

***Theorem* 2.7**

The order of the group $\ker(g)$ *is the order of the direct sum group* $\ker(l) \oplus \pi_0$ $(M^+_{2(p,q)})$. Also, since g is surjective by Lemma 2.3, then we have the following:

***Theorem* 2.8**

T*he order of the group* $\pi_0(M^+_{3(p,q)})$ *is equal to three times the order of the group* $\pi_p SO(q+1) \oplus \pi_q SO(p+1) \oplus \theta^{p+q+1}$.

Proof

Since g is surjective, then $\pi_0(M^+_{3(p,q)}) = \ker(g) \oplus \pi_p SO(q+1)$. But by Lemma 3.6, the order of $\ker(l)$ is equal to the order of $\ker(N)$ and in [1] we showed that order of $\ker(N)$ = order of $\ker(B)$. In [4], Sato showed that order of $\ker(B)$ = order of $\pi_q SO(p+1) \oplus \theta^{p+q}$.

$$\pi_0(M^+_{3(p,q)}) \approx \ker(l) \oplus \pi_0 M^+_{2(p,q)} \oplus \pi_p SO(q+1)$$

But $\ker(l) = \ker(N) = \ker(B) = \pi_q SO(p+1) \oplus \theta^{p+q}$. Thus,

$$\pi_0(M^+_{3(p,q)}) \approx \ker(B) \oplus \pi_0(M^+_{2(p,q)}) \oplus \pi_p SO(q+1).$$

$$= \pi_q SO(p+1) \oplus \theta^{p+q+1} \oplus \pi_0(M^+_{2(p,q)}) \oplus \pi_p SO(q+1)$$

In [1], we proved that the order of $\pi_0(M^+_{2(p,q)})$ is two times the order of $\pi_q SO(q+1)$ $\oplus \pi_p\ SO(p+1) \oplus \theta^{p+q+1}$. Substituting for $\pi_0(M^+_{2(p,q)})$, we have the result order of $\pi_0(M^+_{3(p,q)})$ = three times the order of the group $\pi_p SO(q+1) \oplus \pi_q SO(p+1) \oplus \theta^{p+q+1}$.

4.0 Conclusion

This result therefore extends the result of Sato [4] for $Diff(S^p \times S^q)$ and our result [1] for $Diff\ (S^p \times S^q \# S^p \times S^q)$ to the computation of $Diff\ (S^p \times S^q \# S^p \times S^q \# S^p \times S^q)$.

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