Journal of the Nigerian Association of Mathematical Physics
***Volume* 11 (November 2007), 33 - 38**

On the number of cyclic quotients of some Abelian *p-Groups*

Michael Enioluwafe*
Department Of Mathematics,
University Of Ibadan, Ibadan, Nigeria.

Abstract

We determine in this paper, the precise number of cyclic quotients of Abelian p-groups of exponent p^i and rank r > 1; i = 1 and 2.

1.0 Introduction

The mathematical motivation for this paper is as follows:

Let π be a finite Abelian group, R a commutative Noetherian ring, $G_*(\Lambda)$ the Quillen K-theory of the category of finitely-generated Λ-modules, for any ring Λ with identity. In [4]; D. L. Webb established the formula

$$G_n(Z_\pi) \cong \bigoplus_{\rho \in X(\pi)} G_n(Z \prec \rho \succ), \quad n \geq 0$$

where $Z < \rho >$ denotes the ring of fractions $Z(\rho)[1|\rho|]$ obtained by inverting $|\rho|$, $Z(\rho)$ denotes the quotient of the group ring $Z(\rho)$ by the $|\rho|^{-th}$ cyclotomic polynomial $\Phi_{|\rho|}$ evaluated at a generator of ρ(the ideal factored out is independent of the choice of generator for ρ), $|\cdot|$ denotes cardinality and $X(\pi)$ the set of cyclic quotients of π. A natural problem is that of computing $G_n(Z\pi)$ as explicitly as possible and from the formula above, it is desirable to know the number of cyclic quotients of π. The object of this paper is to establish the precise number of cyclic quotients of π; for $\pi := \underbrace{Z/p^n \oplus \cdots \oplus Z/p^n,}_{r-times} \; n = 1, 2, \; r \succ 1$

The organization of the paper is as follows: Section 2 is devoted to a proof of

T*heorem* A

Let $\underbrace{\pi := Z/p \oplus Z/p \oplus \cdots \oplus Z/p,}_{r-times} \quad r \succ 1, \; p$*, a prime number and γ is a subgroup of π. Then the number of the factor groups π/γ such that |π/γ| = p is* $\dfrac{p^r - 1}{p-1}$.

While in section 3; we shall finally give a proof of

Theorem B

Let $\pi := \underbrace{Z/p^2 \oplus Z/p^2 \oplus \cdots \oplus Z/p^2,}_{r-times} \quad r \succ 1, \; p$ *a prime number and γ ≤ π. Then the number of factor groups π/γ such that |π/γ| =* p^2 *is* $p^{r-1}\left(\dfrac{p^r - 1}{p-1}\right)$.

***Telephone: +2348065765744. Fax Number: 02-8103043. e-mail: maalawo@yahoo.com and mikeaded@ictp.trieste.it A Regular Associate of The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy**

In this paper, we need the following fundamental definition.

***Definition*: (Fundamental)**

Let $\pi := \underbrace{Z/p^i \oplus Z/p^i \oplus \cdots \oplus Z/p^i}_{r-times},\quad i = 1, 2,\ r \succ 1,\ p$*, a prime number and γ a subgroup of π of order* p^{ir-i}*; then we define a subgroup base for γ as* (*r* - *i*)*; r-tuples generating γ. This can be represented as* (*r* - *i*)*-rows of an* $r \times r$*-matrix whose rows generate π.*

2.0 The counting of cyclic quotients of prime order

In this section, we established the following:

***Theorem* A**

Let $\pi := \underbrace{Z/p \oplus Z/p \oplus \cdots \oplus Z/p}_{r-times},\quad r \succ 1,\ p$ *a prime number and γ is a subgroup of π. Then the number of the factor groups π/γ such that |π/γ| = p is* $\frac{p^r - 1}{p - 1}$.

Proof

Let $\pi := \underbrace{Z/p \oplus Z/p \oplus \cdots \oplus Z/p}_{r-times},\quad r \succ 1,\ p$ a prime number.

We define *Z/p* ≅ *Z*p* := <*a*>*;* $\varepsilon_k \in \{a^l\},\ 0 \le l \le p-1$, and applying the fundamental definition given above, we obtain the following set of subgroup base representations in *r* × *r*-matrices:

$$A = \left\{ \begin{pmatrix} a^p & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & \varepsilon_k & 1 & \cdots & 1 & 1 & 1 \\ 1 & a^p & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & a & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & 1 & a^p & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \cdots, \right.$$

$$\left. \begin{pmatrix} a & 1 & 1 & \cdots & \varepsilon_k & 1 & 1 \\ 1 & a & 1 & \cdots & \varepsilon_k & 1 & 1 \\ 1 & 1 & a & \cdots & \varepsilon_k & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a^p & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \cdots & 1 & \varepsilon_k & 1 \\ 1 & a & 1 & \cdots & 1 & \varepsilon_k & 1 \\ 1 & 1 & a & \cdots & 1 & \varepsilon_k & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & \varepsilon_k & 1 \\ 1 & 1 & 1 & \cdots & 1 & a^p & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \cdots & 1 & 1 & \varepsilon_k \\ 1 & a & 1 & \cdots & 1 & 1 & \varepsilon_k \\ 1 & 1 & a & \cdots & 1 & 1 & \varepsilon_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & a & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & 1 & a^p \end{pmatrix} \right\}.$$

Thus, our counting on set ***A*** yields a total sum of cyclic quotients π/γ for which $|\pi/\gamma| = p$ as:

$$1+p+p^2+\cdots+p^{r-3}+p^{r-2}+p^{r-1}.$$

That is, $\frac{p^{r-1}}{p-1}$, for any prime p and any integer > 1. ■

3.0 The counting of cyclic quotients of prime-square order

This section proves the following:

***Theorem* B**

Let $\pi := \underbrace{Z/p^2 \oplus Z/p^2 \oplus \cdots \oplus Z/p^2,}_{r-times}\quad r \succ 1,\ p$ *a prime number and* $\gamma \le \pi$. *Then the number of factor groups* π/γ *such that* $|\pi/\gamma| = p^2$ *is* $p^{r-1}\left(\frac{p^r-1}{p-1}\right)$.

Proof

Let $\pi := \underbrace{Z/p^2 \oplus Z/p^2 \oplus \cdots \oplus Z/p^2,}_{r-times}\quad r \succ 1,\ p$ a prime number. The required cyclic quotients are realized in two cases:

Case 1

We define $Z/p^2 \cong Z^* p^2 := \langle a \rangle,\ \varepsilon_k \in \{a^l\},\ \ 0 \le l \le p^2-1$ and applying the fundamental definition, we form the following set of subgroup base representations in $r \times r$-matrices:

$$B=\left\{\begin{pmatrix} a^{p^2} & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & \varepsilon_k & 1 & \cdots & 1 & 1 & 1 \\ 1 & a^{p^2} & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & a & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & 1 & a^{p^2} & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \cdots,\right.$$

$$\left.\begin{pmatrix} a & 1 & 1 & \cdots & \varepsilon_k & 1 & 1 \\ 1 & a & 1 & \cdots & \varepsilon_k & 1 & 1 \\ 1 & 1 & a & \cdots & \varepsilon_k & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a^{p^2} & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \cdots & 1 & \varepsilon_k & 1 \\ 1 & a & 1 & \cdots & 1 & \varepsilon_k & 1 \\ 1 & 1 & a & \cdots & 1 & \varepsilon_k & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & \varepsilon_k & 1 \\ 1 & 1 & 1 & \cdots & 1 & a^{p^2} & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \cdots & 1 & 1 & \varepsilon_k \\ 1 & a & 1 & \cdots & 1 & 1 & \varepsilon_k \\ 1 & 1 & a & \cdots & 1 & 1 & \varepsilon_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & a & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & 1 & a^{p^2} \end{pmatrix}\right\}.$$

Thus, in this case, we obtain a total sum of cyclic quotients π/γ for which $|\pi/\gamma| = p^2$ as:

$$1 + p^2 + \left(p^2\right)^2 + \cdots + \left(p^2\right)^{r-3} + \left(p^2\right)^{r-2} + \left(p^2\right)^{r-1},$$

which yields the formula: $\dfrac{p^{2r-1}}{p^2 - 1}$. ■

Case 2

In this case, we define $Z/p^2 \cong \{Z_p^*, Z_p^*\}$, $Z_p^* := \langle a \rangle$. This generates a number of sets, namely, $C_1, C_2, \cdots, C_{s-1}, C_3$ of subgroup base representation in $r \times r$-matrices with respect to the definition as:

$$Z/p \cong Z_p^* := \langle a \rangle,$$

$$\varepsilon_\beta \in \{a^i\},\ 1 \le i \le p,\ (i, p) = 1$$

$$\varepsilon_k = \{a^l\},\ 0 \le l \le p-1,$$

and our fundamental definition. So that we can form the set

$$C_1 = \left\{ \begin{pmatrix} a^p & \varepsilon_\beta & 1 & \cdots & 1 & 1 & 1 \\ 1 & a^p & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a^p & 1 & \varepsilon_\beta & \cdots & 1 & 1 & 1 \\ 1 & a & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & 1 & a^p & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \cdots, \begin{pmatrix} a^p & 1 & 1 & \cdots & 1 & 1 & \varepsilon_\beta \\ 1 & a & 1 & \cdots & 1 & 1 & \varepsilon_k \\ 1 & 1 & a & \cdots & 1 & 1 & \varepsilon_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & a & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & 1 & a^p \end{pmatrix} \right\} \text{ and}$$

counting to obtain a sum of cyclic quotients π/γ for which $|\pi/\gamma| = p^2$ as:

$$(p-1) + p(p-1) + \cdots + p^{r-2}(p-1)$$

Next, with similar definitions, we form the set

$$C_2 = \left\{ \begin{pmatrix} a & \varepsilon_k & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & a^p & \varepsilon_\beta & \cdots & 1 & 1 & 1 \\ 1 & 1 & a^p & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & \varepsilon_k & 1 & \cdots & \varepsilon_k & 1 & 1 \\ 1 & a^p & 1 & \cdots & \varepsilon_\beta & 1 & 1 \\ 1 & 1 & a & \cdots & \varepsilon_k & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a^p & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \cdots, \begin{pmatrix} a & \varepsilon_k & 1 & \cdots & 1 & 1 & \varepsilon_k \\ 1 & a^p & 1 & \cdots & 1 & 1 & \varepsilon_\beta \\ 1 & 1 & a & \cdots & 1 & 1 & \varepsilon_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & a & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & 1 & a^p \end{pmatrix} \right\}.$$

Also, counting, we obtain a sum of cyclic quotients π/γ for which $|\pi/\gamma| = p^2$ as:

$$p(p-1)p + p(p-1)p^{r-4} + \cdots + p(p-1)p^{r-2}$$

Continuing with this rule in case 2, we obtain next, with similar definitions applied as above, we have

$$C_{s-1} = \left\{ \begin{pmatrix} a & 1 & 1 & \cdots & \varepsilon_k & \varepsilon_k & 1 \\ 1 & a & 1 & \cdots & \varepsilon_k & \varepsilon_k & 1 \\ 1 & 1 & a & \cdots & \varepsilon_k & \varepsilon_k & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a^p & \varepsilon_\beta & 1 \\ 1 & 1 & 1 & \cdots & 1 & a^p & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \cdots & \varepsilon_k & 1 & \varepsilon_k \\ 1 & a & 1 & \cdots & \varepsilon_k & 1 & \varepsilon_k \\ 1 & 1 & a & \cdots & \varepsilon_k & 1 & \varepsilon_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a^p & 1 & \varepsilon_\beta \\ 1 & 1 & 1 & \cdots & 1 & a & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & 1 & a^p \end{pmatrix} \right\}.$$

and counting gives a sum of cyclic quotients π/γ for which $|\pi/\gamma| = p^2$ as:

$$p^{r-3}(p-1)p^{r-3} + p^{r-3}(p-1)p^{r-2}$$

Finally, following the same rule, we form singleton set

$$C_s = \left\{ \begin{pmatrix} a & \varepsilon_k & 1 & \cdots & 1 & \varepsilon_k & \varepsilon_k \\ 1 & a & 1 & \cdots & 1 & \varepsilon_k & \varepsilon_k \\ 1 & 1 & a & \cdots & 1 & \varepsilon_k & \varepsilon_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & \varepsilon_k & \varepsilon_k \\ 1 & 1 & 1 & \cdots & 1 & a^p & \varepsilon_\beta \\ 1 & 1 & 1 & \cdots & 1 & 1 & a^p \end{pmatrix} \right\}.$$

and counting, we obtain a sum of cyclic quotients π/γ for which $|\pi/\gamma| = p^2$ as:

$$p^{r-2}(p-1)p^{r-2}.$$

Therefore, we obtain a total sum of cyclic quotients from all above sets $C_1, C_2, \cdots, C_{s-1}, C_s$ as

$$(p-1) + p(p-1) + \cdots + p^{r-2}(p-1) + p(p-1)p + p(p-1)p^{r-4} + \cdots + p(p-1)p^{r-2} + \cdots$$
$$+ p^{r-3}(p-1)p^{r-3} + p^{r-3}(p-1)p^{r-2} + p^{r-2}(p-1)p^{r-2},$$

which yields the formula:

$$\frac{p^{r-1} + p^{2r-2} - p^{r+1} - p^{2r-1} + p - 1}{(p^2-1)(p-1)}.$$

Thus, the result of the theorem follows from adding the two cases above, for any prime *p:* and any $r > 1$

■

4.0 Conclusion

This paper solves a very special case of a well-motivated general problem. Further work is in progress to extend the methods and results given here to the general situation.

Acknowledgment

The author would like to thank the ICTP and the SIDA for their generosity and support during 2006 when this work was done.

References

[1] D. Goldfeld, A. Lubotzky, N. Nikolov, L. Pyber, H. Bass, Counting Primes, Groups, and Manifolds, Proceedings of the National Academy of Sciences of the United States of America, Vol 101, 37(Sep., 2004), 13428 - 13430.

[2] A. O. Kuku, Representation theory and higher algebraic K-theory, Pure and Applied Mathematics (Boca Raton), 287. Chapmam and Hall/CRC, Boca Raton, FL, 2007 xxviii + 442pp ISBN: 978-1-58488-603-7;1-58488-603-x.

[3] H. Lenstra, Grothendieck groups of Abelian group rings, J. Pure Appl. Algebra 20(1981), 173-193. 4

[4] J. Shareshian, Topology of subgroup lattices of symmetric and alternating groups, Journal of Combinatorial Theory, series A, 104 (1) (Oct.,2003) 137 - 155.

[5] D. L. Webb, The Lenstra map on classifying spaces and *G*-theory of group rings, invent. Math. 84(1986), 73-89.

[6] D. L. Webb, Higher *G*-theory of nilpotent group rings, J. Algebra 16(1988), 457-465.

[7] D. L. Webb, Quillen *G*-theory of Abelian group rings, J. Pure Appl. Algebra 39(1986), 177-195.