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A new analytical solution to the diffusion problem: Fourier series method

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Abstract

This paper reviews briefly the origin of Fourier Series Method. The paper then gives a vivid description of how the method can be applied to solve a diffusion problem, subject to some boundary conditions. The result obtained is quite appealing as it can be used to solve similar examples of diffusion equations.

***Keywords*:** analytical, diffusion problem, Fourier series.

1.0 Introduction

The Fourier Series Method, developed by Von Neumann during the World War II was first discussed in detail by O'Brien, Hyman and Kaplan in a paper published in 1951.

It expresses an initial line of errors in terms of a finite Fourier Series and considers the growth of a function that reduces to this series for t = 0 by *a* " variables separable" method identical with that commonly used for deriving analytical solution of partial differential equations.

The Fourier series can be formulated in terms of sines or cosines but the algebra is easier if the complex exponential form is used, i.e. with $\sum a_n \cos(nnx/l)$ or $\sum b_n \sin(nnx/l)$ replaced by the equivalent $\sum a_n \ell^{innx/l}$ where $i=\sqrt{-1}$ and l is the interval throughout which the function is defined.

We shall consider the following constrained problems.

2.0 Problem (P1)

$$Min\ \ J(z,u) = \min \int\int \left[u^2(x,t) + z^2(x,t) \right] dx\ dt \tag{2.1}$$

Subject to
$$\frac{\delta z(x,t)}{\delta t} = \frac{\delta^2 z(x,t)}{\delta x^2} + u(x,t)$$
$$z(0,t) = z(l,t) = 0; \quad 0 \le x \le l$$
$$z(x,t) = z_0(x); \quad 0 \le x \le l \tag{2.2}$$

we write the Hamiltonian **H** for (2.1) and (2.2) as

$$H = z^2(x,t) + u^2(x,t) + \lambda^T \left[\frac{\delta^2 z(x,t)}{\delta x^2} + u(x,t) \right] \tag{2.3}$$

where $\lambda^T = \lambda^T(t)$. Setting $f(z,u) = \frac{\delta^2 z}{\delta x^2} + u,\ g(z,u) = z^2 + u^2$ we then have the first order necessary conditions for optimality as

$$\frac{\delta z}{\delta x} = \frac{\delta H}{\delta \lambda} = \frac{\delta^2 z}{\delta x^2} + u = f(z.u) \tag{2.4}$$

$$\frac{\delta \lambda}{\delta t} = -\frac{\delta H}{\delta z} = -\frac{(\delta f)^T}{\delta z \lambda} - \frac{\delta g}{\delta z} = -2z(x,t) \tag{2.5}$$

$$\frac{\delta H}{\delta U} = 0 \text{ or } \frac{(\delta f)^T}{\delta u \lambda} + \frac{\delta g}{\delta u} = 0 \tag{2.6}$$

where $H = g(z,u) + \lambda^t(t) f(z,u)$. Equations (2.6) gives

$$\lambda + 2u = 0 \text{ or } \lambda = -2u \tag{2.7}$$

By virtue of (2.5) and (2.7), we have

$$\frac{\delta \lambda}{\delta t} = -2\frac{\delta u}{\delta t} = -2z$$
$$z(x,t) = \frac{\delta u}{\delta t}(x,t) \tag{2.8}$$

Equation (2.8) is here of physical significance under the conditions for optimality which expresses the relationship between the temperature and the heat source at any point x of the unit conducting rod of our diffusion model. Moreover, we here treat (2.8) as a differential transform of any previously known solution of the diffusion equation. Assuming that (2.8) admits the Fourier solution,

$$Z(x,t) = \sum_{i=1}^{\infty} a_i(t) \sin \pi i x$$
$$U(x,t) = \sum_{i=1}^{\infty} u_i(t) \sin \pi i x \tag{2.9}$$

we then have our new solution $z(x,t) = \frac{\delta}{\delta t} \sum_{i=1}^{\infty} u_i(t) \sin \pi i x$ (2.10)

Hence, it immediately follows that,

$$\alpha_i(t) = u_{it}(t) \;\; (x,t) = \sum_{i=1}^{\infty} u_{i_t}(t) \sin \pi i x \text{ and } z_t(x,t) = \sum_{i=1}^{\infty} u_{it}(t) \sin \pi i x$$
$$Z_{xx}(x,t) = \sum_{i=1}^{\infty} i^2(-\pi^2) u_{it}(t) \sin \pi i x \tag{2.11}$$
$$Z,(x,0) = \sum_{i=1}^{\infty} u_{it}(0) \sin \pi i x$$

Now, problem (P1) becomes

3.0 Problem (P2):

$$Min \int [U^2_1 + u^2_2 + \ldots + u^2_N]\, dt + \int [u^2_{it} + u^2_{2\,t} + \ldots u^2_{N\,t}]\, dt \tag{3.1}$$

Subject to $\frac{\delta z(x,t)}{\delta t} = \frac{\delta^2 z(x,t)}{\delta x^2} + (x,t)$, $z(0, t) = z(l, t) = 0$; $0 \le x \le 1$, $z(x, t) = z_0(x)$; $0 \le x \le 1$.

The corresponding unconstrained problem is given by

$$U_1tt = -\pi^2 1^2 U_{1t} + U_1$$
$$U_{2tt} = -\pi^2 2^2 U_{2t} + U_2$$
$$\vdots \qquad \vdots$$
$$U_{ntt} = -\pi^2 N^2 U_{N_t} + U_N \tag{3.2}$$

4.0 Problem (P3)

$$Min\int\left[U_1^2 + \ldots U_n^2\right]dt + \int\left[U_{1t}^2 + \ldots + U_{Nt}^2\right]dt$$
$$+\mu\int\left[\left\|U_{1tt} + \pi^2 U_{1t} - U_1\right\|^2 + \ldots + \left\|U_{Ntt} + \pi^2 U_{Ntt} - U_N^2\right\|\right]dt \tag{4.1}$$

We proceed to solve (3.2). However, since the sequence of equation (3.2) only differs by constant multiplicands, it suffices to solve just one of the N- second order equations.

Choosing the first equation, we have

$$\frac{\delta^2 u}{\delta t^2} + \frac{\pi^2 \delta u}{\delta t} - u = 0$$
$$\frac{\delta^2 u}{\delta t^2}(.,t) + \frac{\pi^2 \delta u}{\delta t}(.,t) - u(.,) = 0 \tag{4.2}$$

The auxiliary equation for (4.2) is $\lambda^2 + \pi^2\lambda - 1 = 0$, which gives $\lambda = \dfrac{-\pi^2 \pm \sqrt{\pi^4 + 4}}{2}$. Thus ,

$U(.,t) = C_1\ell^{\lambda_1 t} + C_2\ell^{\lambda_2 t}$, where $\lambda_1 = \dfrac{-\pi^2 + \sqrt{\pi^4 + 4}}{2}$, $\lambda_2 = \dfrac{-\pi^2 - \sqrt{\pi^4 + 4}}{2}$. So, we have

$$U(.,t) = C_1 \exp\left[\frac{-\pi^2 + \sqrt{\pi^4 + 4}}{2}\right]t + C_2 \exp\left[\frac{-\pi^2 + \sqrt{\pi^4 + 4}}{2}\right]t \tag{4.3}$$

$\dfrac{du}{dt} = \lambda_1 C_1 \ell^{\lambda_1 t} + \lambda_2 C_2 \ell^{\lambda_2 t}$, It is not difficult to see that

$$U(.,t) = u(0) = C_1 + C_2 = \sum_{i=1}^{\infty} u_1(0)\sin \pi i x$$

and $$\frac{du}{dt}(.,t) = u_t(0) = \lambda_1 C_1 + \lambda_2 C_2 = \sum_{i=1}^{\infty} u_{1t}(0)\sin \pi i x.$$

From the last two equations, it becomes obvious that

$$C_2 = \frac{1}{\lambda_1 - \lambda_2}\left[\sum_{i=1}^{\infty}\lambda_1 u_i(0) - u_{1t}(0)]\sin \pi i x\right] \text{ and } C_1 = \sum_{i=1}^{\infty} u_i(0)\sin \pi i x - C_2$$

$$= \sum_{i=1}^{\infty} u_i(0)\sin \pi ix - \frac{1}{(\lambda_1 - \lambda_2)}\left\{\sum_{i=1}^{\infty}[\lambda_1 u_i(0) - u_{it}(0)]\sin \pi ix\right\}$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left\{-\lambda_2 \sum_{i=1}^{\infty} u_i(0)\sin \pi ix + \sum_{i=1}^{\infty} u_{it}(0)\sin \pi ix\right\}$$

By virtue of the expression for C_1 and C_2 in (4.3), we have

$$u(x,t) = \frac{1}{(\lambda_1 - \lambda_2)}\left\{\sum_{i=1}^{\infty} u_{it}(0)\sin \pi ix - \lambda_2 \sum_{i=1}^{\infty} u_{it}(0)\sin \pi ix\right\}\exp(\lambda_1 t)$$
$$+ \frac{1}{(\lambda_1 - \lambda_2)}\left\{\sum_{i=1}^{\infty}[\lambda_1 u_i(0) - u_{it}(0)]\sin \pi ix\right\}\exp(\lambda_2 t) \qquad (4.4)$$

But $\lambda_1 - \lambda_2 = \frac{\left\{-\pi^2 + \sqrt{\pi^4 + 4}\right\}}{2} - \frac{\left\{-\pi^2 - \sqrt{\pi^4 + 4}\right\}}{2} = \sqrt{\pi^2 + 4}$. Thus, we have

$$U(x,t) = \frac{\ell^{\lambda_1 t}}{\sqrt{\pi^4 + 4}}\left\{\sum_{i=1}^{\infty}[u_{it}(0) - \lambda_2 u_i(0)]\sin \pi ix\right\}$$
$$+ \frac{\ell^{\lambda_2 t}}{\sqrt{\pi^4 + 4}}\left\{\sum_{i=1}^{\infty}[\lambda_1 u_i(0) - u_{it}(0)]\sin \pi ix\right\} \qquad (4.5)$$

and

$$z(x,t) = \frac{\delta u}{\delta t}(x,t) = \frac{\lambda_1 \ell^{\lambda_1 t}}{\sqrt{\pi^4 + 4}}\left\{\sum_{i=1}^{\infty}[u_{it}(0) - \lambda_2 u_i(0)\sin \pi ix]\right\} + \frac{\lambda_2 \ell^{\lambda_2 t}}{\sqrt{\pi^4 + 4}}\left\{\sum_{i=1}^{\infty}[\lambda_1 u_i(0) - u_{it}(0)]\sin \pi ix\right\} \qquad (4.6)$$

Explicitly written out, we have

$$Z(x,t) = -\left\{\sum_{i=1}^{\infty}\left[u_i(0)\frac{\left[-\pi^2 - \sqrt{\pi^4 + 4} - u_{it}(0)\right]\sin \pi ix}{2}\right]\right\}\left\{\frac{\left[-\pi^2 - \sqrt{\pi^4 + 4}\left[1^2\right]\right]}{2\sqrt{\pi^2 + 4}}\,\frac{\left[\exp - -\pi^2 - \sqrt{\pi^4 + 4}\right]t}{2}\right\}$$

$$+ \frac{\{1\}}{\sqrt{\pi^4 + 4}}\left[\exp\frac{\left[-\pi^2 - \sqrt{\pi^4 + 4}\right]t}{2}\;\frac{\left[-\pi^2 - \sqrt{\pi^4 + 4}\right]}{2}\right]\left\{\sum_{i=1}^{\infty} u_i(0)\frac{\left[-\pi^2 - \sqrt{\pi^4 + 4}\right]}{2} - u_{it}(0)\sin \pi ix\right\} \text{ or}$$

equivalently as in the next equation:

$$z(x,t)=\frac{1}{\sqrt{\pi^4+4}}\frac{\left[\exp-\pi^2-\sqrt{\pi^4+4}\right]t}{2}\frac{\left[-\pi^2-\sqrt{\pi^4+4}\right]}{2}\left\{\sum_{i=1}^{\infty}\left[u_i(0)\frac{\left[-\pi^2-\sqrt{\pi^4+4}\right]}{2}-u_{it}(0)\right]\sin\pi ix\right\}$$

$$-\frac{\left[\exp\left(-\pi^2-\sqrt{\pi^4+4}\right]t}{2}\frac{\left[-\pi^2+\sqrt{\pi^4+4}\right]}{2}\left\{\sum_{i=1}^{\infty}\left[u_i(0)\frac{\left[-\pi^2+\sqrt{\pi^4+4}\right]}{2}u_{it}(0)\right]\sin\pi ix\right\} \qquad (4.7)$$

and $$U(x,t)=\frac{1}{\sqrt{\pi^4+4}}\left[\frac{\exp\left(-\pi^2-\sqrt{\pi^4+4}\right)}{2}\right]t\left\{\left[\sum_{1=i}^{\infty}\left[u_{it}(0)-\frac{\left[-\pi^2-\sqrt{\pi^4}+4\right]}{2}\right]\right]\right\}$$

$$-u_i(0)[\sin\pi ix]+\left[\frac{\exp\left(-\pi^2-\sqrt{\pi^4=4}\right)}{2}\right]t\}$$

$$\left\{\left[\sum_{i=1}^{\infty}[u_i(0)\frac{\left[\pi^2+\sqrt{\pi^4+4}\right]}{2}-u_{it}(0)]\sin\pi ix]\right\} \qquad (4.8)$$

Equation (4.7) shows that $Z(0, t) = Z(1, t) = 0$, which satisfies the boundary conditions and

$$Z=(x,0)=Z_0(x)=-\left\{-\frac{\left(\pi^2+\sqrt{\pi^4+4}\right)}{2\sqrt{\pi^4+4}}\sum_{i=1}^{\infty}[u_i(0)\frac{\left[-\pi^2-\sqrt{\pi^4+4}\right]}{2}-u_{it}(0)]\sin\pi ix]\right\}$$

$$-\left\{\frac{\left[-\pi^2-\sqrt{\pi^4+4}\right]}{2\sqrt{\pi^4+4}}\sum_{i=1}^{\infty}[u_i(0)\sum_{i=1}\infty\left[\frac{\left[-\pi^2+\sqrt{\pi^4+4}\right]}{2}-u_{it}(0)\right]\sin\pi ix]\right\}$$

$$=\left\{\frac{\left[-\pi^2-\sqrt{\pi^4+4}\right]}{2\sqrt{\pi^4+4}}\sum_{i=1}^{\infty}\left[u_i(0)\frac{\left[-\pi^2+\sqrt{\pi^4-4}\right]}{2}-u_{it}]\sin\pi ix]\right]\right\}$$

$$\left\{\frac{\left[-\pi^2+\sqrt{\pi^4+4}\right]}{2\sqrt{\pi^4+4}}\sum_{i=1}^{\infty}\left[u_i(0)\frac{\left[-\pi^2-\sqrt{\pi^4+4}\right]}{2}-u_{it}\,]\sin \pi ix]\right\} \tag{4.9}$$

5.0 Conclusion

In this paper, the Fourier Series Method has been discussed. The diffusion problem that occurs frequently in physical and engineering applications is given. With this present contribution, new vistas of research are opened for further application of Fourier Series Method to justify that the Von Neumann's Method is a well established integral transform.

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