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The direct product of right zero semigroups and certain groupoids

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Abstract

This paper investigates first the structure of semigroups which are direct products of right zero semigroups and cancellative semigroups with identity. We consider the relationship of these semigroups to right groups (the direct products of groups and right zero semigroups). Finally, we consider groupoids which are direct products of right singular semigroups and unipotent (one-idempotent) groupoids with identity.

***Keywords*:** Decomposition, partition, band of, orthogonal, cancellative, idempotent.

1.0 Introduction

An *M*-group *S* is a groupoid [4] which satisfies the following conditions:

P.1.1. *S* contain a left identity

P.1.2. If y or z is a left identity of *S*, then $(xy)z = x(yz)$ for all x in *S*.

P.1.3. For any *s* in *S*, there is a unique left identity *e* (which may depend on *x*) such that $xe = x$

Such systems were investigated by Tamura, Merkel and Latiemr in [2].

We will follow the notation and terminology of Clifford and Preston [1] for all concepts not defined in this paper.

2.0 Left cancellative semigroups and right groups

A decomposition of a groupoid *S* is a partition of *S*, $S = U(S_a : a \text{ in } A)$, $S_a \cap S_b = \Pi$, $a \neq b$, in which for every a,b in *A*, there is a *c* in *A* such that $S_a S_b \subseteq S_c$ [2]. If for every such $a,b,c = b$ and each S_a is a groupoid of type *T*, we say *S* is a right zero band of groupoids of type *T*. Semigroups which are "bands of semigroups of type *T*" have been studied extensively [1]. A right group [1] is a semigroup *S* such that if *a,b* in *S* there exists a unique *s* in *S* such that $ax = b$. An *M*-groupoid which is a semi group is called an *M*-semigroup.

We now give a different formulation of a lemma stated in [2].

***Lemma* 2.1** [2].

A groupoid (semigroup) S is isomorphic to the direct product of two groupoids (semigroups) A and B if and only if there exists homomorphisms U *and* V *onto A and B respectively such that for a in A and b in B,* $aU^{-1} \cap bV^{-1} = (x_{ab})$, *a set consisting of exactly one element. In this case U and V are said to be orthogonal.*

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Proof

If x in $aU^{-1} \cap bV^{-1}$, let $xW = (a,b)$. It is easily seen that W is the required isomorphism. The converse is clear.

For the purpose of illustrating this important principle of "orthogonality of decompositions" [2], we will give a proof of the following result established in [2]. Our proof using Lemma 2.1 is slightly different from that given in [2].

Theorem 2.2 [2]

An M-groupoid (semigroup) S is the direct product of a right zero semigroup and a groupoid (semigroup) with identity and conversely. The groupoid (semigroup) with identity is obtained as Se, where e is a left identity.

Proof

We first show that if *e* is a left identity of *S*, S_e is a groupoid with identity *e*. If *se, te in Se*, then

$$(se)(te) = ((se)\ t)\ e \text{ in } Se \text{ and } (se)e = s\,(ee) = se.$$

Let *R* denote the subset of left identities of *S*. Clearly, *R* is a right zero semi-group. If *s* in *S*, let *e* denote the unique left identity of P.1.3.

Let $xU = xe$ and $xV = e_x$ for *x* in *S*. Since $(xe)(ye) = x(e(ye)) = x(ye) = (xy)\ e, U$ is a homomorphism of *S* onto. *Se*. Since $(xy)\ e_{xy} = xy = x(ye_y) = (xy)\ e_y$,

$$e_{xy} = e_y \text{ and } e_{xy} = e_x e_y$$

If e in R, $e = e_e$. . Thus, V is a homomorphism of *S* onto *R*. Let *xe* and g be arbitrary elements of *Se* and *R* respectively. If $zU = xe$ and $zV = g$, that is, *ze = xe* and *zg = z*, then

$$z = zg = z\,(eg) = (ze)\,g = (xe)\,g = x\,(eg) = xg$$

However, $(xg)e = x\,(ge) = xe$ and $(xg)g = xg$. Thus, *U* and *V* are orthogonal and *S = RxSe* by Lemma 2.1. The converse follows by routine calculation.

***Theorem* 2.3**

The following conditions on a semigroup S is equivalent.

(1) *S is left cancellative; ba = ca implies bx = cx for all x in S:a in aS for every a in S.*

(2) $S = E \times C$ *where E is a right zero semigroup and C is a cancellative semigroup with identity.*

(3) *S is a right zero bond of cancellative semigroups with identity.*

Proof

The proof of this theorem is cyclic.

(1) $\rightarrow$ (2): We first show that *S* is an *M*-semigroup. Let *E* denote the set of idempotents of *S*. If *a* in *S*, there exists *e* in *S* such that *a = ae*. Hence $ae = ae^2$, e^2, and *e* in *E*, that is, $E. \neq \Pi$. If *e* in *E* and *a* in *S*,

$$ea = e^2a = e(ea), \text{ and } ea = a$$

Thus, every idempotent is a left identity. Next, let *g* be a fixed element of *E*, and let us consider *Sg*. Clearly, *Sg* is left cancellative. Suppose that $(ug)(sg) = (vg)(sg)$ with s,u,v in *S*. Thus $ug = (ug)g = (vg)g = vg$ and *Sg* is cancellative. Apply Theorem 2.2.

(2) $\rightarrow$ (3). For *g* in *E*, let $Sg = ((g,c) : c \text{ in } C)$. Clearly, *Sg* is a cancellative semigroup with identity $[g,1]$ where 1 is the identity of *C* and *S* is a right zero band of the *Sg*.

(3) $\rightarrow$ (1). Let *S* be a right zero band of the semigroup $(S_t : t \text{ in } T)$ where S_t is a cancellative semigroup with identity e_t. If *x* in S_t y in S_w, and *a* in S_u, where t, *u*, *w* in *T* and *ax = ay*, then *w = t* and $a(e_t x) = a(e_t y)$. . Thus, $(ae_t)x = (ae_t)y$. and *x = y* since *x,y* and ae_t in S_t. Hence, *S* is left cancellative. If *xa = ya*, $x(e_u a) = y(e_u a)$, $(xe_u)a = (ye_u)\ a$, and $xe_u = ye_u$. If *z* in S_v (*v* in *T*),

$$e_u z = (e_u e_v)z = e_u(e_u z) \text{ and } z = e_u z.$$

Thus, $xz = x(e_u z) = (xe_u)z = (ye_u)z = y\,(e_u z) = yz.$

Corollary 2.4

The following assertions concerning a semigroup are equivalent.

(1) *S is a right group*

(2) $S = G \times E$ *where G is group and E is a right zero semigroup*

(3) *S is a right zero band of groups*

Proof

(1) $\longrightarrow$ (2). If *ba* =*ca* (*a*, *b*, *c* in *S*) and *x* in *S* choose *y* in *S* such that *ay* = *x*. Thus, *bay* = *cay* and *bx* = *cx*. Similarly, *a* in *aS*. Thus, by Theorem 2.3 $S = C \times E$ where *C* is a cancellative semigroup with identity 1 and *E* is a right semigroup. Thus if $(a,h),(1,h)$ in $C \times E$, there exists (*b*, *t*) in $C \times E$ such that $(a,h)(b,t) = (1,h)$, that is, *ab* = 1.

(2) $\longrightarrow$ (3). The proof is similar to the proof (2) $\longrightarrow$ (3) in Theorem 2.3.

(3) $\longrightarrow$ (1) Suppose *S* is the right zero band of groups $(S_a : a \text{ in } A)$. If *x* in S_a, *y* in $S_b(a,b \text{ in } A)$, *xy* and *y* in S_b and hence there exits *z* in S_b such that $x(yz) = (xy)z = y$. *S* is left cancellative by Theorem 2.3 The equivalence of (1) and (2) is given by [1].

3.0 M-groupoids.

A groupoid is called unipotent (one-idempotent) if it contains precisely one idempotent [1]. An idempotent *e* of a groupoid *S* is said to be primitive if *f* is an idempotent of *S* such that *ef* = *fe* = *f*, then *e* = *f*. Primitive idempotents have played an important role in algebra (see [1], for example).

In [3], a groupoid *S* is said to satisfy $\bigcup$ (notation of [3]) if it verifies the following conditions:

P.3.1. Existence of a left identity.

P.3.2. $(xy)z = x(yz)$ if *y* or *z* is a left identity.

P.3.3. If *e*, *f* are indepotents, *ef* = *f*.

P.3.4. There is a decomposition $(S_a : a \text{ in } A)$ of *S* such that S_a is a groupoid with two-sided identity e_a.

We will say that a groupoid *S* satisfies $\cup$, (notation of [3]) if P.3.1, P.3.2, P.3.4 hold and if *S* verifies.

P.3.3)'. If *a*, *b* in *A*, $e_a e_b = e_b$.

***Theorem* 3**.1

$\bigcup_2'$ *Characterizes an M-groupoid* [3].

Proof

For the proof of sufficiency, we may show that $\bigcup_2'$, implies P.1.3. If *e* is a left identity of *S*, *e* in *Sa* for some *a* in *A* and hence $e = e_a$. Next, we show that each e_a is a left identity of *S*. Let e_b be a left identity of *S* (the existence of such is guaranteed by P.3.1). If *x* is an arbitrary element of *S*,

$$e_a x = e_a(e_b x) = (e_a e_b)x = e_b x = x$$

If *x* in S_a say, $xe_a = x.$. If $xe_c = x.$ with $c \neq a$, $S_a S_c \subseteq S_a$ and $e_c = e_a e_c$ in S_a, a contradiction. Thus $e_c = e_a$ and P.1.3 is valid.

To prove necessity, we note that *S* is the direct product of a right zero samigroup *E* and a groupoid *G* with identity 1 by Theorem 2.2. If for *e* in *E*, we let $S_e = ((e,g) : g \text{ in } G)$, (S_e) is a decomposition of *S* such that each S_e is a groupoid with identity (*e*,1). The collection of left identities is ((*e*, 1) in *E*). It is now a routine calculation to establish P.(3.3)`.

***Theorem* 3.2**

The following conditions on a groupoid are equivalent.

(1) *S is an M-groupoid such that each left identity is a primitive idempotent.*

(2) *S* = *E* $\times$*G, where E a right is zero semigroup and G is an unipotent groupoid with identity.*

(3) *S satisfies* P.1.1 *and* P.1.2 *and S is a right zero band of unipotent groupoid with identity.*

(4) *S satisfies* $\bigcup_3'$.

Proof

(1) $\longrightarrow$ (2). Let *e* be a left identity of *S* and suppose that *g* is an idempotent of *Se*. Then, *g* = *xe* for some *x* in *S*, $ge = (xe)e = x(ee) = xe = g$, .and *e* = *g*. Apply Theorem 2.2.

(2) $\longrightarrow$ (3). It is easy to see that ((*e*, 1): *e* in *E*), where 1 is the identity of *G*, is both the collection of idempotents and the collection of left identities of $E \times G$.. P.1.2 is verified by routine calculation. If $S_e = ((e,g): g$ in $G)$, *S* is a right zero band of the (S_e) and each S_e is clearly a unipotent groupoid with identity.

(3) $\longrightarrow$ (4). Clearly, P.3.1, P.3.2, and P.3.4 are valid. Let *S* be the right zero band $(S_a : a$ in A) of unipotent groupoids with identity e_a.

Clearly $(e_a : a$ in *A*) is the collection of idempotents of *S*. Let e_a be a left identity of *S* and suppose e_b is an arbitrary idempotent of *S*.

If $e_b e_a = z$ in S_a, $e_b z = e_b(e_b e_a) = e_b e_a = z$ -and $z^2 = (e_b e_a)z = e_b\,(e_a z) = e_b z$ = *z*, that is $e_b e_a = e_a$. If *c* in *A*, $e_b e_c = e_b\,(e_a e_c) = (e_b e_a)e_c = e_a e_c = e_c$ and P.3.3 is verified.

(4) $\rightarrow$ (1). Clearly, $\bigcup_3'$ implies $\bigcup_2'$.. Thus, *S* is *M*-groupoid by Theorems 3. If *e* and *f* are idempotent and *ef* = *fe* = *f*, then *e* = *f* P.3.3.

If the semigroup *S* is the right zero band $(S_a : a\ in\ A)$ of unipotent samigroup with identity e_a,

$$(e_a e_b)(e_a e_b) = e_a\,(e_b(e_a e_b)) = e_a\,(e_a e_b) = e_a e_b \text{ and } e_a e_b = e_b(a,b,\ in\ A).$$

If x in S_c, say $e_a x = e_a\,(e_c x) = (e_a e_c)x = e_c x = x$ and e_a is a left identity. Thus, the modification of Theorem 3.2 to semigroups is clear.

We next now give the analogue of Theorem 2.3 for groupoids.

***Theorem* 3.3**

The following assertions on a groupoid are equivalent.

(1) *S satisfies* P.1.1 and P.1.2. *S is left cancellative; ba = ca implies bx = cx for all x in S; if a in S; there exists a left identity e such that a = ae..*

(2) $S = E \times C$, *where E is a right zero semigroup and C is a cancellative groupoids with identity.*

(3) S *satisfies* P.1.1 *and* P.1.2 *and S is a right zero band of cancellative groupoids with identity*

Proof

(1) $\rightarrow$ (2). Clearly, *S* is an *M*-groupoids. If *g* is a left identity of *S*, we show as in the proof of "(1) $\rightarrow$ (2)." in Theorem 2.3, that *Sg* is cancellative. Apply Theorem 2.1.

(2) $\longrightarrow$ (3). Since *C* is unipotent, we proceed as in the proof of "(2) $\longrightarrow$ (3). " in Theorem 3.2.

(3) $\longrightarrow$ (1). Let *S* be the right zero band of cancellative groupoids with identity $(S_a : a$ in $A)$ where e_a is the identity of S_a. By Theorem 3.2, $e_a e_b = e_b(a,b$ in $A)$ and P.1.1, P.1.2, and P.1.3 are valid.

4.0 Conclusion

We have shown that e_a is a left identity of an *M*-group *S*, and also that *M*-groupoid *S* is the director product of a right zero semigroup, that is, a semigroup, such that $xy = y$ for *x*, *y*, and is, a groupoid with identity.

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