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On compactoid and limited sets in non-Archimedean locally convex spaces

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Abstract

In [2] and [3] spaces in which every bounded subset is a compactoid was studied. Every compactoid set is limited but the converse is not true [3]. In this paper, we shall study some spaces in which every limited set is compactoid.

***Keywords*:** Controllability, nonlinear neutral systems, distributed delays, time varying multiple lumped delays.

1.0 Introduction

Let *E* be a linear space. If *E* has a linear topology we call E a Linear Topological Space (l. t. s.). If in addition the linear topology on E is Hausdorff we call E a Hausdorff Linear Topological Space. If an l. t. s.

E has a base of neighbourhoods at the origin consisting of convex sets we call E a locally convex space. A Hausdorff l. t. s. which has a base of neighbourhoods at the origin consisting of convex sets is called a Hausdorff locally convex space. It is interesting to mention that not every l. t. s. is locally convex. For Example L^p, $0 < p < 1$, the space of continuous functions for which $\int |f|_p d\mu < \infty$ is not locally convex.

Let *K* be a field with a non- archimedean valuation $|.|$. That is the valuation $|.|$. on K satisfies the additional condition: $|x + y| \leq \max \{ |x|, |y| \} \ \forall x,y \in K$.

Clearly this valuation induces on K an ultrametric d because it satisfies the additional condition: $d(x,y) \leq \max \{d(x,z), d(z,y)\}$

$\forall x,y,z \in K$. *K* together with d is also called a **non - Archimedean valued field**. The ultrametric valued field *K* has some striking interesting properties like the ones in the following result:

***Theorem* 1.1:**

Let K be an ultrametric valued field and let $x,y,z \in K$ and $r > 0$ define: $S_r(x) = \{z \in k: d(x,z) < r\}$ and $C_r(x) = \{z \in k: d(x,z) \leq r\}$. Then:

(i) If $d(a,b) < d(b,c)$ then $d(a,c) = d(b,c)$.

(ii) If $b \in S_r(a)$, then $S_r(a) = S_r(b)$. If $b \in C_r(a)$ then $C_r(a) = C_r(b)$.

(iii) The set $\{x \in X: d(a,x) = r\}$ is open and closed in X.

(iv) $\text{Sup} \{d(x,y): x,y \in C_r(a)\} \leq r$ and $\text{Sup} \{d(x,y): x,y \in S_r(a)\} \leq r$

Proof

See [2], [5].

In this paper *K* will denote a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $|.|$. *E* and *F* will denote Hausdorff locally convex spaces.

2.0 Compactoid sets, limited sets and GP-spaces

***Definition* 2.1**

A subset *B* of *E* is called **Compactoid** if for every neighbourhood U in *E* $\exists$ a finite subset S of *E* such that $B \subset coS + U$ where coS denotes the absolutely convex hull of *S*.

Clearly every compactoid set is bounded, but the converse is not necessarily true. But there are spaces in which every bounded set is compactoid. [2], [3].

***Definition* 2.2**

A bounded subset B of E is said to be **Limited** in E, if every equicontinuous $\sigma(E,{}^{n}E)$ null sequence in E^{I} converges to o uniformly on B [5].

Now it is obvious that every compactoid subset of E is limited in E. Our interest here is to attempt to study spaces in which every limited set is compactoid, such spaces are called **Gelfand – Phillips Spaces** (GP–Spaces) following Lindstrom and Schlumprecht who studied such spaces in the complex case [5].

Now we known that there is a natural identification of the $\sigma(E',E)$ - null sequences in E^{I} with the continuous linear maps from E to C_0 ([1], Lemma 2.2). If we combine this with the form of the compactoid subsets of C_0 ([8], proposition 2.1). We obtain the following:

***Proposition* 2.3**

A bounded subset B of E is limited in E if and only if for each continuous linear map T from E to C_0, T(B) is compactoid in C_0. The following results follow easily:

***Proposition* 2.4**

(i) Every compactoid subset of E is limited in E

(ii) If B is limited in E and $T\in L(E,F)$ then T(B) is limited in F where L(E,F) denotes the vector space of all continuous linear maps from E to F.

(iii) If B is limited in E and $D \subset B$ then D is limited in E.

(iv) Let M be a subspace of E and $B \subset M$. If B is limited in M then B is bounded in E. The converse is also true when M is complemented or dense in E for an example showing that the converse is not generally true. It follows from proposition 2.3 that if every continuous linear map from E to C_0 is compact, then every bounded subset of E is limited. In particular, if the valuation on K is dense, we have the following:

***Corollary* 2.5**

If the valuation on *K* is dense, then the unit ball of L^{∞} is limited (Non-compactoid) in L^{∞}.

***Remark* 2.6**

Corollary 2.5 shows that for densely valued fields the behaviour of limited sets in non-Archimedean functional analysis is in sharp contrast with the one of locally convex spaces over the real or complex field. We will see in theorem 2.8 that this difference is even more striking when the valuation on *K* is discrete.

***Definition* 2.7** (Comp [6])

A locally convex space is called a **Gelfand - Phillips Space** (GP – Space in short) if every limited set in E is compactoid. The following results are obvious:

***Preposition* 2.8**

(i) A subset of a GP - space is a GP - space.

(ii) The product of a family of GP - spaces is a GP - space.

***Theorem* 2.9**

(i) Every locally convex space E of countable type is a GP - space.

(ii) Every Banach space with a base is a GP - space.

(iii) If the valuation on K is discrete then every locally convex space over K is a GP - space.

Proof

(i) From proposition 2.3, if follows that C_0 [and hence every normed space of countable type] is a GP - space. Using the fact that E can be considered as a subspace of p

$\prod_{p\in p} E_p$

where *P* is a family of seminorms determining the topology of E and for each $p \in p$ E_p is the normed space. E/ker p

Now all the Ep are of countable type we apply (2.8)

(ii) Let $A \subset C$ be limited. Let us assume that A is absolutely convex. It suffices to show that every countable subset B of A is compactoid. Let [B] denote the closed linear hull of B. Then ([7], Corollary 3 (8) [B]) is complemented in E and so by composition (2.4)(iv) we have that B is limited in [B]. By [1] B is compactoid in [B] and hence in E.

(iii) Using the fact that $E \subset \prod_{p\in p} E_p$ where now each of the spaces Ep has a base ([7], theorem 5.16) then applying (ii) and proposition 2.8 the result follows.

***Remark* 2.9**

Property (iv) of proposition 2.4 is not true in general, e.g., Let the valuation on K be dense, take $E = L^\infty$, $M = C_0$ and B the unit ball in E. Then apply corollary 2.5, proposition (2. (i) and proposition 2.5 (i).

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