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Relative controllability of nonlinear neutral systems with distributed and multiple lumped delays in control

R. A. Umana
Department of Mathematics and Computer Sciences
Federal University of Technology, Owerri, Nigeria.

Abstract

In this paper we study the relative controllability of nonlinear neutral system with distributed and multiple lumped time varying delays in control. Using Schauder's fixed point theorem sufficient conditions for relative controllability in a given time interval are formulated and proved.

***Keywords*:** Controllability, nonlinear neutral systems, distributed delays, time varying multiple lumped delays.

1.0 Introduction

Controllability is one of the fundamental concepts in mathematical control theory. This is a qualitative property of dynamical control systems and is of particular importance in control theory. Controllability is the property of being able to steer between two arbitrary points in the state space.
In recent years various controllability problems for different types of nonlinear dynamical systems with different types of delays in control have been studied by several authors [1-10]. For instance, Balachandran and Anandhi [6] studied the controllability of neutral functional integrodifferential infinite delay systems in Banach spaces by using the analytic semigrouped and the Nussbaum fixed point theorem. Park and Kang [8] derived a set of sufficient conditions for the approximate controllability of neutral functional differential system with unbounded delay. Onwuatu [3] investigated the controllability problem of nonlinear systems with distributed delays in the control. Klamka [2] discussed the controllability of semilinear systems with multiple delays in control by using a generalized open mapping theorem. Recently in [7] Balachandran et al obtained the existence results for nonlinear abstract neutral differential equations with time varying delays and as an application the controllability problem for the neutral system is discussed.

However, it should be stressed, that the most literature in this direction has been mainly concerned with controllability problems for nonlinear dynamical systems with either time varying multiple delays, distributed delays or infinite delays in the control. Not many studies are undertaken to investigate the controllability of nonlinear dynamical systems with both distributed and time varying multiple delays in control.

The present endeavour therefore is to develop sufficient conditions for the relative controllability of nonlinear neutral functional differential systems with both distributed and time varying multiple lumped delays in control. This is motivated by the fact that models for systems with delay in the control occur in population studies and in some complex economic systems. Specifically, models for systems with distributed delays in the control occur in the study of agricultural economics and population dynamics [5]. Our results generalize and extend known results to systems with both distributed and multiple delays in the control.

2.0 Preliminaries

Suppose $R=(-\infty,\infty), R^n$ is an n-dimensional real vector space with norm $|.|$. If $J=[t_0,t_1]$ is any interval of R, the usual Lebesgue space of square integrable (equivalence, classes of) functions from J to R^n will be denoted by $L_2(J,R^n)$. We let $C=C([-h,0],R^n)$, and designate the norm of an element ϕ in C by $\|\phi\|=\sup_{-h\le s\le 0}|\phi(s)|$.

Consider the nonlinear neutral systems with distributed and multiple time-varying delays of the form

$$\frac{d}{dt}D(t,x_t) = L(t,x_t)+\int_{-h}^{0}d_\theta B(t,\theta,x(t),u(t))u(t+\theta)$$
$$+ f(t,x(t),u(w_0(t)),u(w_1(t),...,u(w_i(t)),...,u(w_N(t))),\ t\in[t_0,t_1]$$

$$x_{t0}=\phi \qquad (2.1)$$

where $x\in R^n$, u is an m-dimensional control function with $u\in C_m[t_0-h,t_1],\ B(t,\theta,x,u)$ is an $n\times m$ dimensional matrix, continuous in (t,x,u) fixed for θ and of bounded variation in θ on $[-h,0]$ for each $(t,x,u)\in J\times R^{n+m}$. Assume the n-dimensional vector function f is continuous in its arguments. The integral is in the Lebesgue-Stieltjes sense which is denoted by the symbol d_θ. The continuous strictly increasing functions $w_i(t):[t_0,t_1]\to R,\ i=0,1,2,...,N,$ represent deviating arguments in the control, that is, $w_i(t)=t-h_i(t)$, where $h_i(t)$ are lumped time varying delays for $i=0,1,2,...,N$.

Let $h>0$ be given. For a function $x:[t_0-h,t_1]\to R^n$ and $t\in[t_0,t_1]$, we use the symbol x_t to denote the function on $[-h,0]$ defined by $x_t(s)=x(t+s)$ for $s\in[-h,0]$. Similarly, for a function $u:[t_0-h,t_1]\to R^m$ and $t\in[t_0,t_1]$ we use the symbol u_t to denote the function on $[-h,0]$ defined by $u_t(s)=u(t+s)$ for $s\in[-h,0]$.

Suppose $L,D:R\times C\to R^n$ are given functions. We say that a relation of the type;

$$\frac{d}{dt}D(t,x_t)=L(t,x_t) \qquad (2.2)$$

is a functional differential equation of neutral type. A function x is said to be a solution of (2.2) if there exists $t_0\in R, a>0$ such that $x\in C([t_0-h,\ t_0+a],R^n), t\in(t_0,t_0+a)$ and x satisfies (2.2) on $[t_0,t_0+a]$. Given $t_0\in R,\phi\in C$, we say $x(t_0,\phi)$ is a solution of (2.2) with initial value (t_0,ϕ) if there exists an $a>0$ such that $x(t_0,\phi)$ is a solution of (2.2) on $[t_0-h,t_0+a]$ and $x_{t0}(t_0,\phi)=\phi$.

In system (2.2), assume that $D(.,.):R\times C\to R^n$ is defined by $D(t,x_t)=x(t)-g(t,x_t)$, where $g(t,\phi), L(t,.)$ are bounded linear operators from C into R^n for each fixed t in $[0,\infty], g(t,\phi)$ is continuous for

$$(t,\phi)\in[0,\infty)\times C, g(t,\phi)=\int_{-h}^{0}d_s\mu(t,s)\phi(s), L(t,\phi)=\int_{-h}^{0}d_s\eta(t,s)\phi(s), |g(t,\phi)|\le k\|\phi\|,$$

$|L(t,\phi)|\le L(t)\|\phi\|,\ (t,\phi)\in[0,\infty)\times C$ for some nonnegative constant k, continuous nonnegative l and $\mu(t,.),\eta(t,.)$ are $n\times n$ matrix functions of bounded variation on $[-h,0]$.

With the condition on (2.1), solutions of (2.1) exist and are unique and depend continuously on initial conditions. Furthermore if the solution $x_t(t_0,\phi,0)$ of (2.2) is designated by $T(t,t_0)\phi$, the variation of constant formula yields the existence of an $n\times n$ matrix function $H(t,s,x,u)$ defined for $0\le s\le t+h, t\in[0,\infty)$, continuous in s from the right, of bounded variation in s, $H(t,s,x,u)=0,\ t<s\le t+h$, such that the solution $x(t_0,\phi,u,f)$ of (1) is given by $x_{t0}(t_0,\phi,u,f)=\phi$

and

$$x(t,t_0,u,f) = T(t,t_0)\phi(0) + \int_{t_0}^{t} H(t,s,x,u)\left(\int_{-h}^{0} d_\theta B(s,\theta,x,u)\, u(s+\theta)\right)ds$$

$$+ \int_{0}^{t} H(t,s,x,u)\; f(s, x(s), u(w_0(s)), u(w_1(s)), \ldots, u(w_N(s)))ds \qquad (2.3)$$

Using the unsymmetric Fubini theorem as in [10], equation (2.3) can be written as

$$x(t,t_0,u,f) = T(t,t_0)\phi(0) + \int_{-h}^{0} d_{B\theta} \int_{t_0+\theta}^{t_0} H(t,s-\theta,\theta,x,u) B(s-\theta,\theta,x,u) u_{t_0}(s)ds$$

$$+ \int_{t_0}^{t}\left(\int_{-h}^{0} H(t,s-\theta,x,u) d_\theta B_t\,(s-\theta,\theta,x,u)\right)u(s)ds$$

$$+ \int_{t_0}^{t} H(t,s,x,u)\; f(s, x(s), u(w_0(s)), u(w_1(s)), \ldots, u(w_N(s)))ds \qquad (2.4)$$

where $d_{B\theta}$ denotes that the integration is in the Lebesgue-Stieltjes sense with respect to the variable θ in B and

$$B_t(s,\theta,x,u) = \begin{cases} B(s,\theta,x,u), & for\;\; s \le t \\ 0, & for\;\; s > t \end{cases} \qquad (2.5)$$

Define

$$q(t,x,u_{t_0}) = \int_{-h}^{0} d_{B\theta} \int_{t_0+\theta}^{t} H(t,s-\theta,x,u)\, B(s-\theta,\theta,x,u)\, u_{t_0}(s)ds \qquad (2.6)$$

$$Z(t,s,x,u) = \int_{-h}^{0} H(t,s-\theta,x,u)\, d_\theta B_t(s-\theta,\theta,x,u) \qquad (2.7)$$

and the controllability matrix

$$W(t_0,t,x,u) = \int_{t_0}^{t} Z(t_1,s,x,u)\, Z^T\,(t,s,x,u)ds \qquad (2.8)$$

where T denotes the matrix transpose.
The functions $w_i(t):[t_0,t_1] \to R, i = 0,1,2,\ldots,N$ are twice continuously differentiable and strictly increasing in $[t_0,t_1]$. Moreover, $w_i(t) \le t$ for $t \in [t_0,t_1]$, and $i = 0,1,2,\ldots,N$.

***Definition* 2.1**

The set $z(t) = \{x(t), x_t, u_t\}$ *is said to be the complete state of the system* (2.1) *at time* t.

***Definition* 2.2**

The system (2.1) *is said to be relatively controllable on* J *if, for every initial complete state* $z(t_0)$ *and every vector* $x_1 \in R^n$, *there exists a control function* $u(t)$ *defined on* $[t_0,t_1]$ *such that the solution of the system* (2.1) satisfies $x(t_1) = x_1$.

3.0 Main Result

We are now ready to obtain the main result of this paper. For this, we will take

$$p = (x, x', u_0, u_1, \ldots, u_i, \ldots, u_N) \in R^m \times R^n \times R^{(N+1)m}$$

and let $|p| = |x| + |x'| + |u_0| + |u_1| + \ldots + |u_i| + \ldots + |u_N|$.

Theorem 3.1

Let the continuous function f *satisfy the so called growth condition*

$$\lim_{|p|\to 0} \frac{|f(t,p)|}{|p|} = 0 \qquad (3.1)$$

uniformly in $t \in J$, and suppose that the functions g and L are continuous in their argument and continuously differentiable with respect to argument x.
Furthermore, we assume that there exists a positive constant d such that for each pair of functions $(x,u) \in C_n[t_0,t_1] \times C_m[t_0,t_1]$

$$\det W(t_0,t_1,x,u) \ge d\,. \qquad (3.2)$$

Then the system (2.1) is relatively controllable on J .

Proof

Let

$$F = C_n[t_0, t_1] \times C_m[t_0, t_1]$$

and define the nonlinear continuous operator

$$T : F \to F \text{ by } T(x,u) = (y,v),$$

where

$$v(t) = Z^T(t_1, t, x, u)W^{-1}(t_0, t_1, x, u)[x_1 - T(t_1, t_0)\phi(0) - q(t_1, x, u_{t_0})$$
$$- \int_{t_0}^{t} H(t_1, s, x, u)\, f(s, x(s), u(w_0(s)), u(w_1(s)), ..., u(w_N(s)))ds] \quad (3.3)$$

for $t \in J$, and $v(t) = 0$ for $t \in [-h, 0]$;

$$y(t) = T(t_1, t_0)\phi(0) + q(t, x, u_{t_0}) + \int_{t_0}^{t} Z(t, s, x, u)\, v(s)ds$$
$$+ \int_{t_0}^{t} H(t_1, s, x, u)\, f(s, x(s), u(w_0(s)), u(w_1(s)), ..., u(w_N(s)))ds] \quad (3.4)$$

for $t \in J$, and $y(t) = \phi(t)$ for $t \in [-h, 0]$.

Let us introduce the following notations:

$a_1 = \sup\{|Z(t_1, t, x, u)| : (x,u) \in F, t \in J\}$

$a_2 = \sup\{|W^{-1}|(t_0, t_1, x, u) : (x,u) \in F, t \in J\}$

$a_3 = \sup\{|T(t_1, t_0)\phi(0)| + |q(t_1, x, u_{t_0})| : (x,u) \in F, t \in J\} + |x_1|$

$a_4 = \sup\{|H(t, s, x, u)| : u \in C_m[t_0, t_1]\}$

$a_5 = \sup\{|f(t, x(t), u(w_0(t)), u(w_1(t)), ..., u(w_N(t)))| : (x,u) \in F, t \in J\}$

$b = \max\{(t_1 - t_0)a_1, 1\}$

$c_1 = 6ba_1a_2a_4(t_1 - t_0)$

$c_2 = 6a_4(t_1 - t_0)$

$c_3 = (t_1 - t_0)a_1$

$d_1 = 6a_1a_2a_3b$

$d_2 = 6a_3$

$c = \max\{c_1, c_2\}$

$d = \max\{d_1, d_2\}$

Then, $|v(t)| \le a_1a_2[a_3 + a_4a_5(t_1 - t_0)] \le b\|v\|_{c_m(t_1,t_0)} + \frac{1}{6}(d + ca_5)$

and

$$|y(t)| \le a_3 + (t_1 - t_0)a_1\|v\|_{c_m(t_1 - t_0)} + (t_1 - t_0)a_4a_5 \le \frac{1}{6b}(d + ca_5) \quad (3.5)$$

It follows from the growth condition (9) that there exists a positive constant r such that, if $|p| \le r$ then

$$d + c|f(t, p)| \le r, \text{ for all } t \in [t_0, t_1] \quad (3.6)$$

Let us define the set Q as follows:

$$Q = \{(x,u) \in C_n(t_0,t_1) \times C_m(t_0,t_1) : \|x\|_{c_n(t_0,t_1)} \le r(N+2)^{-1}$$

and

$$\|u\|_{c_m(t_0,t_1)} \le r(N+2)^{-1}\}$$

Thus, if we take

$$\|x\|_{c_n(t_0,t_1)} \le r(N+2)^{-1},$$

$$\|u\|_{c_m(t_0,t_1)} \le r(N+2)^{-1}$$

and moreover,

$\|u_{t_0}\|_{c_m(t_0,t_1)} \le r(N+2)^{-1}$ then

$$|p| = |x(t)| + \sum_{i=0}^{N} |u(w_i(t))| \le r \text{ for all } t \in [t_0,t_1] \qquad (3.7)$$

Thus we have proved that the nonlinear operator T maps Q into itself. Since all the functions involved in the definition of the operator T are continuous, it follows that T is continuous. Hence by the application of the Arzela-Ascoli theorem T is completely continuous. Since the set Q is closed, bounded, and convex, the Schauder fixed point theorem guarantees that T has a fixed point $(x,u) \in Q.$ It follows that

$$x(t) = T(t_1,t_0)\phi(0) + q(t,x,u_{t_0}) + \int_{t_0}^{t} Z(t,s,x,u)ds + \int_{t_0}^{t} H(t,s,x,u) f(s,s(s),u(w_0(s)),u(w_1(s)),...,u(w_N(s)))ds$$

Hence $x(t)$ is a solution of the system (2.1). Moreover, it is easy to verify that this solution satisfies the final condition $x(t_1) = x_1$. Hence (2.1) is relatively controllable on J .

4.0 Conclusion

In the present paper sufficient conditions for the relative controllability of nonlinear neutral functional differential systems with distributed and multiple lumped time varying delays in the control have been formulated and proved. In the proof of the main results well known Schauder's fixed point theorem has been used. The method presented in this paper is in fact quite general and covers wide classes of nonlinear dynamical control systems. Therefore, similar relative controllability results may be derived for more general class of nonlinear dynamical control systems.

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