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First order normalization in the perturbed restricted three–body problem with variable mass

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Abstract

This paper performs the first order normalization that will be employed in the study of the nonlinear stability of triangular points of the perturbed restricted three – body problem with variable mass. The problem is perturbed in the sense that small perturbations are given in the coriolis and centrifugal forces. It is with variable mass as the mass of the third body varies with time. It is found that these perturbations and varying mass are capable to bring a change in the Lagrangian function, and consequently in the basic frequencies. They become successful in affecting the angle coordinates but remain unsuccessful in changing the action momenta coordinates. The transformation utilized for reduction of the second order part of the Hamiltonian to the normal form is also dependent on the perturbed basic frequencies.

***Keywords*:** Normalization, Perturbed, RTBP with variable mass,

1.0 Introduction

The main problem in modern dynamics is the long-time prediction of the state of mechanical systems that have been modeled by differential or functional equations. The emphasis is no longer on the computational of individual solutions but on characterization of classes such as stable or unstable manifolds of solutions. For Hamiltonian systems, it is not enough to look at the linearized system alone. The higher order terms in the normalized systems can affect the stability.

In order to investigate the stability of an equilibrium point for all time and all the orders of the terms in the expansion of the Hamiltonian one has to apply KAM theorem [the work of Kolmogrov [1] extended by Arnold [2] and moser [3]]. For this, Hamiltonian H is to be reduced to the normalized Hamiltonian form, $H=\omega_1 I_1-\omega_2 I_2+\frac{1}{2}\left(AI_1^2+2BI_1I_2+CI_2^2\right)+\cdots$ with ω_1, ω_2 as basic frequencies, I_1, I_2 as the action momenta coordinates: and A, B, C as second order coefficients in the frequencies. Here H_2 = $\omega_1 I_1-\omega_2 I_2$. If H_2 is of positive definite form, the equilibrium position is stable by virtue of Liapunov's [4] theorem for all orders and all time. On the other hand if H_2 is not a function of definite sign, then the investigation of stability needs KAM theorem. To apply KAM theorem one must need normalization of the Hamiltonian.

Hence, the aim of this paper is to perform the first order normalization that will be utilized in the study of the nonlinear stability of triangular points of the present problem.

Deprit and Deprit [5] investigated the nonlinear stability of triangular points of the classical restricted three-body problem. Bhatnagar and Hallan [6] studied the effect of perturbations in the coriolis and centrifugal forces respectively on the nonlinear stability of equilibrium points of the above problem. Further, the nonlinear stability in the restricted three-body problem under different aspects was also discussed by Niedzielska [7], Subbarao and Sharma [8] and Gozdziewski [9].

This paper is organized as follows. In section 2 we determine the Lagrangian of the problem. Then it is expanded in power series of ξ and η, where (ξ,η) are the coordinates of the third body referred to the triangular point L_4 as the origin. Arranging the terms in ascending powers of (ξ,η) the homogeneous poly-nomial L_2 of order

2 is found. Section 3 establishes the relation between the perturbed and the unperturbed basic frequencies. With L_2 Lagrange's equations of motion are written and then under linear stability condition characteristic roots are obtained. The normal form of the second order part of the Hamiltonian is presented in section 4, where we solve a set of linear equations obtained from H_2. under the normality conditions, we apply the transformation defined by Whittaker [10] for reducing H_2 to the normal form. In section 5 we are with conclusion.

2.0 Expansion of Lagrangian

The equations of motion of the restricted three-body problem, under the assumption that the mass of the third body varies with time, as found by Singh and Ishwar [11], can be written as $\xi'' - 2\phi\eta' = \Omega_{\xi}, \ \eta'' + 2\phi\xi' = \Omega_\eta$ with

$$\Omega = \left(\frac{\beta^2}{4}+1\right)\frac{\psi}{2}\left(\xi^2+\eta^2\right)+\gamma^{3/2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right),\ r_1^{\,2} = \left(\xi+\mu\gamma^{\frac{1}{2}}\right)^2+\eta^2,$$

$$r_2^2 = \left\{\xi-(1-\mu)\gamma^{\frac{1}{2}}\right\}^2+\eta^2,\ \beta = \alpha\, m_0^{n-1},\ \gamma = \frac{m}{m_0}, \qquad \alpha = \text{constant},\ 0.4 \le n \le 4.4.$$

The mass m of the third body varies with the time t. m_o is its mass at $t = 0$. Primes indicate differentiation with respect to Γ where dt = $\gamma^{-k}\, d\Gamma$. The parameter μ is the ratio of the mass of the smaller primary to the total mass of the primaries such that $0<\mu \le ½$. Perturbation in the coriolis and the centrifugal forces have been considered with the aid of ϕ and ψ such that the unperturbed value of each is unity. Consequently they may be taken as $\phi = 1+\varepsilon,\ |\varepsilon| << 1 \quad and \quad \psi = 1+\varepsilon',\ |\varepsilon'| << 1$, where ε and ε' represent the perturbations in the coriolis and the centrifugal forces respectively. The coordinates of triangular equilibrium point L_4, as found in Singh and Ishwar [11], are

$\left(\frac{\gamma_1}{2},\frac{\gamma_2}{2}\right)$, where $\gamma_1 = (1-2\mu)\gamma^{\frac{1}{2}}$, $\gamma_2 = \gamma^{\frac{1}{2}}\left[4\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}\left(1-\frac{2}{3}\varepsilon'\right)-1\right]^{\frac{1}{2}}$.The perturbed Lagrangian function of the restricted problem with variable mass can be written as

$$L = \frac{1}{2}\left(\xi'^2+\eta'^2\right)+\phi\left(\xi\eta'-\eta\xi'\right)+\frac{\psi}{2}\left(1+\frac{\beta^2}{4}\right)\left(\xi^2+\eta^2\right)+\gamma^{\frac{3}{2}}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right),$$

We shift the origin to the triangular point L_4. For that we change

$\xi \to \xi+\frac{\gamma_1}{2},\ \ \eta \to \eta+\frac{\gamma_2}{2},$. Then the Lagrangian function L becomes

$$L = \frac{1}{2}\left(\xi'^2+\eta'^2\right)+\left[\left(\xi+\frac{\gamma_1}{2}\right)\eta'-\left(\eta+\frac{\gamma_2}{2}\right)\xi'\right](1+\varepsilon)+\frac{1}{2}\left[\left(\xi+\frac{\gamma_1}{2}\right)^2+\left(\eta+\frac{\gamma_2}{2}\right)^2\right]$$
$$\left(\frac{\beta^2}{4}+1\right)(1+\varepsilon')+\gamma^{3/2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right), \tag{2.1}$$

where $r_1^{-1} = \left[\left(\xi+\frac{\gamma^{1/2}}{2}\right)^2+\left(\eta+\frac{\gamma_2}{2}\right)^2\right]^{-1/2} = f(\xi,\eta)$ (say),

$$r_2^{-1} = \left[\left(\xi-\frac{\gamma^{1/2}}{2}\right)^2+\left(\eta+\frac{\gamma_2}{2}\right)^2\right]^{-1/2} = g(\xi,\eta) \quad \text{(say).}$$

In order to expand L in power series of ξ and η we first of all expand $f(\xi,\eta)$ and $g(\xi,\eta)$ by Taylor's theorem. Then we put the values of r_1^{-1} and r_2^{-1} in (1) and arrange the terms in ascending powers of ξ, η. Since we need here only L_2, so we have

$$L_2 = \frac{1}{2}\left(\xi'^2+\eta'^2\right)+\left(\xi\eta'-\xi'\eta\right)(1+\varepsilon)+\frac{1}{8}\left(3+5\varepsilon'\right)\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}\xi^2+$$

$$+\frac{1}{8}\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}\left\{\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}(12+12\varepsilon')-(3+5\varepsilon')\right\}\eta^2+\frac{1}{4\sqrt{3}}\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}(3+5\varepsilon')$$

$$\left\{4\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}(3-2\varepsilon')-3\right\}^{\frac{1}{2}} s\xi\eta \text{ , where } s = 1 - 2\mu \tag{2.2}$$

3.0 Relation between perturbed and unperturbed basic frequencies

We wish to establish relation between perturbed and unperturbed basic frequencies. For this to the first order, we write Lagrange's equations of motion

$$\xi''-2(1+\varepsilon)\eta'=\frac{1}{4}\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}(3+5\varepsilon')\left[\xi+\frac{1}{\sqrt{3}}\left\{4\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}(3-2\varepsilon')-3\right\}^{\frac{1}{2}}s\eta\right]$$

and

$$\eta''+2(1+\varepsilon)\xi'=\frac{1}{4}\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}\left[\frac{1}{\sqrt{3}}(3+5\varepsilon)\left\{4\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}(3-2\varepsilon')-3\right\}^{\frac{1}{2}}s\xi+\right.$$

$$\left.+\left\{\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}(12+12\varepsilon')-(3+5\varepsilon')\right\}\eta\right]$$

The characteristic equation of these equations is given by

$$\lambda^4+\left[(1+8\varepsilon-3\varepsilon')-\frac{3}{4}\beta^2(1+\varepsilon')\right]\lambda^2$$

$$+\frac{3}{16}(1-s^2)\left(\frac{\beta^2}{4}+1\right)^{\frac{10}{3}}\left[4\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}(3+8\varepsilon')-(3+10\varepsilon')\right]=0. \tag{3.1}$$

Let η_c denote the critical value of the mass parameter which depends upon the varying mass of the third body and small perturbations in the coriolis and centrifugal forces.

We have seen that for $0 \prec \mu_c \prec \mu_c 0$, the roots of (3.1) are distinct and pure imaginary and the triangular point L_4 is stable in the linear sense [12]. Let these roots be $\pm i\,\omega_1'$, $\pm i\ \omega_2'$, where ω_1', ω_2', represent the perturbed basic frequencies. When $\varepsilon = \varepsilon' = 0,$ the values of $\varepsilon, \varepsilon'$ represent unperturbed basic frequencies. We can write

$$\omega_1' = \omega_1(1+p\varepsilon+p'\varepsilon') \text{ and } \omega_2' = \omega_2(1+q\varepsilon+q'\varepsilon') \tag{3.2}$$

where ω_1', ω_2' represent unperturbed basic frequencies such that $\omega_1^2+\omega_2^2 = 1-\frac{3}{4}\beta^2$,

$$\omega_1^2\omega_2^2=\frac{9}{16}(1-s^2)\left(\frac{\beta^2}{4}+1\right)^{\frac{10}{3}}\left[4\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}-1\right], 0<\omega_2<\frac{1}{\sqrt{2}}<\omega_1<1$$

and $p=-q=\frac{4}{k^2}$, $k^2=2\omega_1^2-1+\frac{3}{4}\beta^2=1-2\omega_2^2-\frac{3}{4}\beta^2$, $p'=\dfrac{22\omega_1^2-49-\frac{4}{9}\beta^2\omega_1^2+\frac{367}{36}\beta^2}{18k^2}$

$$q'=-\frac{22\omega_2^2-49-\frac{4}{9}\beta^2\omega_2^2+\frac{367}{36}\beta^2}{18k^2} \tag{3.3}$$

The relations between the perturbed and unperturbed basic frequencies are given by (3.2) whereas p, p', q, q' and k are determined by (3.4).

4.0 Normal form of second order part of Hamitonian

As the normalization procedure depends strongly on the form of H_2 terms, thus it is important to simplify it as much as possible. For this purpose, the Hamiltonian H corresponding to the Lagrangian function L given by (1.1) can be written as

$$H = -L + P_\xi \xi' + P_\eta \eta' \qquad (4.1)$$

where P_ξ, P_η are the momenta coordinates given by $P_\xi = \frac{\partial L}{\partial \xi'}$ and $P_\eta = \frac{\partial L}{\partial \eta'}$. Applying the translation

$\xi \to \xi + \frac{\gamma_1}{2}$, $\eta \to \eta + \frac{\gamma_2}{2}$, $p_\xi \to p_\xi - \phi\frac{\gamma_2}{2}$, $p_\eta \to p_\eta + \phi\frac{\gamma_1}{2}$ on (4.1), one can obtain a new form of H. Expanding the new form of H in power series of ξ and η, the second order part of the Hamiltonian can be expressed as

$$H_2 = \frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) + (1+\varepsilon)(\eta p_\xi + \xi p_\eta) + \left\{\frac{1+8\varepsilon-5\varepsilon'}{8} - \frac{3(3\varepsilon'+1)\beta^2}{32}\right\}\xi^2 +$$

$$\left\{\frac{-5+8\varepsilon-7\varepsilon'}{8} - \frac{(7\varepsilon'+24)\beta^2}{32}\right\}\eta^2 - \frac{s}{4\sqrt{3}}\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}(3+5\varepsilon')\left\{4\left(\frac{\beta^2}{4}+1\right)^{-\frac{2}{3}}(3-2\varepsilon')-3\right\}^{\frac{1}{2}}\xi\eta$$

We solve a set of equations obtain from

$$-\lambda P_\xi = \frac{\partial H_2}{\partial \xi}, \qquad -\lambda P_\eta = \frac{\partial H_2}{\partial \eta}, \qquad \lambda\xi = \frac{\partial H_2}{\partial P_\xi}, \qquad \lambda\eta = \frac{\partial H_2}{\partial P_\eta}.$$

Substituting the four values of λ, namely $\lambda_1 = i\omega_1'$, $\lambda_2 = i\omega_2'$, $\lambda_3 = -i\omega_1'$, $\lambda_4 = -i\omega_2'$ we obtained four values of each ξ, η, p_ξ, p_η and if they are denoted by $\xi_i, \eta_i, (p_\xi)_i, (p_\eta)_i$ (i = 1, 2, 3, 4), we get

$$\xi_i = K_i(-d + 2a\lambda_i \quad \eta_i = K_i(2b + \lambda_i^2 - a^2), \quad (P_5)_i = K_i\left(a\lambda_i^3 - \lambda_i d - 2ab + a^3\right),$$

$$(P_\eta)_i = K_i\left(\lambda_i^3 + a^2\lambda_i + 2b\lambda_i - ad\right),$$

where a = $1+\varepsilon$, b = $\frac{1+8\varepsilon-5\varepsilon'}{8} - \frac{3}{32}(3\varepsilon'+1)\beta^2$, c = $\frac{-5+8\varepsilon-7\varepsilon'}{8} - \frac{(7\varepsilon'+24)\beta^2}{32}$,

$d = -\frac{s}{4\sqrt{3}}\left(\frac{\beta^2}{4}+1\right)^{5/3}(3+5\varepsilon')\left[4\left(\frac{\beta^2}{4}+1\right)^{-2/3}(3-2\varepsilon')-3\right]^{1/2}$ and K_i (i = 1, 2, 3, 4) are constant of proportionality satisfying the normality conditions:

$$\xi_1(P_\xi)_3 - \xi_3(P_5)_1 + \eta_1(P_\eta)_3 - \eta_3(P_\eta)_1 = 1, \quad \xi_2(P_\xi)_4 - \xi_4(P_5)_2 + \eta_2(P_\eta)_4 - \eta_4(P_\eta)_2 = 1 \qquad (4.2)$$

Now, we apply the canonical transformation from the phase space (ξ, η, p_ξ, p_η) into the phase space product of the angle coordinates (θ_1, θ_2) and the action momenta (I_1, I_2) as in Whittaker [10],

$$\begin{pmatrix}\xi\\ \eta\\ P_\xi\\ P_\eta\end{pmatrix} = A'\begin{pmatrix}Q_1\\ Q_2\\ P_1\\ P_2\end{pmatrix}, \qquad (4.3)$$

where $A' = (a'_{ij})_{1\le i,j\le 4}$, $Q_i = \left(\frac{2I_i}{\omega_i'}\right)^{\frac{1}{2}}\sin\theta_i$, $P_i = (2I_i\omega_i')^{\frac{1}{2}}\cos\theta_i\,(i=1,2)$, the different elements of the dydic A' are $a'_{ij} = a_{ij}(1+\alpha_{ij}\varepsilon+\alpha'_{ij}\varepsilon')$, $i, j = 1,2,3,4$ where $a_{11} = 0$, $a_{12} = 0$, $a_{13} = \frac{l_1}{2k\omega_1}$,

$a_{14} = -\frac{l_2}{2k\omega_2}$, $a_{21} = -\frac{4\omega_1}{kl_1}$, $a_{22} = -\frac{4\omega_2}{2kl_2}$, $a_{23} = -\frac{3\sqrt{3}s}{2kl_1\omega_1}$, $a_{24} = \frac{3\sqrt{3}s}{2kl_2\omega_2}$, $a_{31} = -\frac{m_1\omega_1}{2kl_1}$, $a_{32} = -\frac{m_2\omega_2}{2kl_2}$,

$a_{33} = \frac{3\sqrt{3}s}{2kl_1\omega_1}$, $a_{34} = -\frac{3\sqrt{3}s}{2kl_2\omega_2}$, $a_{41} = -\frac{3\sqrt{3}s\omega_1}{2kl_1}$, $a_{42} = -\frac{3\sqrt{3}s\omega_2}{2kl_2}$, $a_{43} = \frac{n_1}{2kl_1\omega_1}$,

$$a_{44} = -\frac{n_2}{2kl_2\omega_2},\ \alpha_{1j} = \alpha'_{1j} = 0,\ (j = 1,2),\ \alpha_{13} = -\frac{\left[88\omega_1^2 + \left(3\omega_1^4 - \frac{1223}{12}\omega_1^2 - \frac{2813}{16}\right)\beta^2\right]}{k^4 l_1^2},$$

$$\alpha'_{13} = \frac{\left[-176\omega_1^6 - 364\omega_1^4 + 1332\omega_1^2 - 135 + \frac{736}{9}\beta^2\omega_1^6 + \frac{15380}{3}\beta^2\omega_1^4 - \frac{24649}{4}\beta^2\omega_1^2 + 51750\beta^2\right]}{18k^4 l_1^2}$$

$$\alpha_{21} = \frac{\left[16\omega_1^6 + 20\omega_1^4 + 24\omega_1^2 - 63 + \left(\frac{6377}{48} - \frac{2911}{12}\omega_1^2 - \frac{33}{2}\omega_1^4\right)\beta^2\right]}{k^4 l_1^2},$$

$$\alpha'_{21} = \frac{\left[-176\omega_1^6 - 76\omega_1^4 - 324\omega_1^2 + 621 - \left(1000\omega_1^6 - \frac{2516}{9}\omega_1^4 - \frac{4769}{12}\omega_1^2 + \frac{565}{8}\right)\beta^2\right]}{18k^4 l_1^2},$$

$$\alpha_{23} = -\frac{8\left[\left(8\omega_1^4 + 7\omega_1^2\right) - \left(\frac{203}{72}\omega_1^4 - \frac{523}{288}\omega_1^2 + \frac{3519}{16}\right)\beta^2\right]}{k^4 l_1^2},$$

$$\alpha'_{23} = \frac{\left[-176\omega_1^6 + 532\omega_1^4 + 740\omega_1^2 - 63 + \left(\frac{637}{3}\omega_1^6 - \frac{23077}{36}\omega_1^4 - \frac{16939}{16}\omega_1^2 - \frac{204997}{192}\right)\beta^2\right]}{18k^4 l_1^2},$$

$$\alpha_{31} = \frac{\left[44\omega_1^4 + \omega_1^2 - 27 + \left(\frac{33}{2}\omega_1^4 + \frac{17}{3}\omega_1^2 - 11\right)\beta^2\right]}{k^4 l_1^2 m_1},$$

$$\alpha'_{31} = \frac{\left[704\omega_1^8 + 864\omega_1^6 - 5924\omega_1^4 + 1656\omega_1^2 + 1755 + \left(913\omega_1^8 + 753\omega_1^6 + \frac{2873}{3}\omega_1^4 - \frac{4652}{5}\omega_1^2 + 228\right)\beta^2\right]}{18k^4 l_1^2 m_1}$$

$$\alpha_{33} = 1 + \alpha_{23},\ \alpha'_{33} = \alpha'_{23},\ \alpha_{41} = \alpha_{23} + 2p,\ \alpha'_{41} = \alpha'_{23} + 2p',$$

$$\alpha_{43} = \frac{-\left[64\omega_1^8 - 64\omega_1^6 + 492\omega_1^4 + 540\omega_1^2 - 81 + \left(23\omega_1^8 + 74\omega_1^6 - \frac{385}{3}\omega_1^4 - \frac{1109}{4}\omega_1^2 + 210\right)\beta^2\right]}{18k^4 l_1^2 n_1}$$

$$\alpha'_{43} = \frac{\left[704\omega_1^8 - 2432\omega_1^6 + 4644\omega_1^4 + 6480\omega_1^2 - 1215 + \left(658\omega_1^8 + 1735\omega_1^6 - 1188\omega_1^4 + \frac{8113}{4}\omega_1^2 - 937\right)\beta^2\right]}{18k^4 l_1^2 n_1}$$

$$l_i^2 = 4\omega_i^2 + 9 - \frac{77}{6}\beta^2,\ m_i = 4\omega_i^2 + 1 - \beta^2,\ n_i = -4\omega_i^2 + 9 - \beta^2,\ (i = 1,2)$$

The values of α_{ij} and α'_{ij} for $j = 2,4$ can be obtained from those for $j=1,3$ respectively by replacing ω_1 by ω_2, l_1 by l_2, m_1 by m_2 n_1 by n_2 wherever they occur, keeping k unchanged.

The transformation (4.3) changes the second order part of the Hamiltonian into the normal form

$$H_2 = \omega'_1 I_1 - \omega'_2 I_2.$$

The general solution of the corresponding equations of motion are $I_i = const\ .(i = 1,2)$,

$$\theta_1 = \omega'_1\Gamma + const\ .\ \theta_2 = -\omega'_2 + const\ .$$

5.0 Conclusion

We infer that small perturbations ε and ε' given in the coriolis and centrifugal forces respectively, and the variation of mass of the third body affect the Lagrangian basic frequencies and angle coordinates. But they do not influence the action momenta coordinates. The transformation utilized for reducing the second order part of the

References

[1] Kolmogrov, A. N. : 'General theory of Dynamical systems', proc. of the 1954 Intern. Congress Math, P. 315 – 333, North Holland, Amsterdam (1957).

[2] Arnold, V. I.: 'On the stability of positions of equilibrium of Hamiltonian system of ordinary differential equations in the general elliptic case', Dokl. Akad Nauk USSR 137, P. 255 and also soviet math 2, P. 247 (1961).

[3] Moser, J.: 'On invariant curves of area preserving mappings of an annulus', Nach. Akad. Wiss. Göttingern, Math. Phys. K. I, II, P. 1 – 20 (1962).

[4] Liapunov, A. M.: 'A General Problem of stability of motion', Acad. Sc. USSR (1956).

[5] Deprit, A. and Deprit – Bartholome A.: 'Stability of the triangular Lagrangian points', Astron. J. 72, (2), P. 173 – 179 (1967).

[6] Bhatnagar, K. B. and Hallan, P.P.: 'The effect of perturbations in coriolis and centrifugal forces on the nonlinear stability of equilibrium points in the restricted problem of three bodies'. Celest. Mech. 30 (2), P. 97 – 114 (1983).

[7] Niedzielska, Z.: 'Nonlinear stability of the libration points in the photogravitational restricted three-body problem', Celest. Mech. and Dyn. Astron. 58 (3), P. 203 – 213 (1994).

[8] Subba Rao, P. V. and sharma, R. K.: 'Effect of oblateness on the nonlinear stability of L_4 in the restricted three body problem', Celest. Mech. & Dyn. Astron. 65 (3), P. 291 – 312 (1997).

[9] Gozdziewsk, K.: 'Stability of the triangular libration points in the unrestricted planar problem of a symmetric rigid body and a point mass', Celest. Mech. & Dyn. Astron. 85 (1), P. 79 - 103 (2003).

[10] Whittaker, E. T. : 'A tretise on the Analytical Dynamics of Particles and Rigid Bodies', Cambridge Uni. Press. London (1965)

[11] Singh, J. and Ishwar, B.: 'Effect of perturbations on the location of equilibrium points in the restricted problem of three bodies with variable,' Celest. Mech. 32 (4), P. 297 – 305 (1984).

[12] Singh, J. and Ishwar, B.: 'Effect of perturbations on the stability of triangular points in the restricted problem of three bodies with variable mass,' Celest. Mech. 35 (3), P. 201 – 207 (1985).