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Relative controllability of nonlinear neutral systems with multiple delays in state and control

R. A. Umana
Department of Mathematics and Computer Sciences
Federal University of Technology, Owerri, Nigeria.

Abstract

Sufficient conditions are developed for the relative controllability of nonlinear neutral systems with time-varying multiple delays in both state and control. The results are obtained by using Schauder's fixed-point theorem.

1.0 Introduction

The primary motivation for the study of neutral functional differential equations is the application to transmission-line theory. It is known that the mixed initial-boundary hyperbolic partial differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral differential equation [11].
The problem of controllability of neutral functional differential equations has received much attention in recent years. Balachandran et al. [10] studied the controllability problem for nonlinear functional differential systems in Banach spaces. Park and Han [7] derived a set of sufficient conditions for the controllability of nonlinear functional integrodifferential systems in Banach Spaces whereas in [5] Han et al investigated the controllability problem of integrodifferential systems by considering the initial condition in some abstract phase space. Balachandran and Sakthivel [2] discussed the controllability of neutral functional integrodifferential systems in Banach Spaces by using the semigroup theory and the Schaefer fixed point theorem. Recently in [6] Fu studied the controllability and local controllability of abstract neutral functional differential systems with unbounded delay whereas in [4] Balachandran and Anandhi established a set of sufficient conditions for controllability of neutral functional integrodifferential infinite delay systems in Banach Spaces. More recently, Umana and Nse [8] studied the null controllability of nonlinear integrodifferential systems with delays in the state and control variables.

The main purpose of this paper is to extend the results of [12], that is, inspired by the recent work in [4, 6], we will study the relative controllability of the following nonlinear neutral functional differential systems with time varying multiple delays in state and control:

$$\frac{d}{dt}D(t,x_t)=\sum_{i=0}^{p}A_i(t)x(t-h_i)+\sum_{i=0}^{p}B_i(t)u(t-h_i)+f(t,x(t),x(t-h),u(t),u(t-h))$$
$$x(t)=\phi(t),\ t\in[-h,0].$$

Equations of this type have applications in the study of electrical networks containing lossless transmission lines; electrodynamics; variational problems, etc [13]. Our approach, similar to one used by Do [9] for nonlinear neutral systems, is to define the appropriate control and its corresponding solution by an integral equation. This equation is then solved by applying the Schauder fixed-point theorem. The results extend those of [12] to neutral systems.

2.0 Preliminaries

Let n and m be positive integers, R the real line $(-\infty,\infty)$. Denote by R^n, the space of real n -

tuples with the Euclidean norm defined by $\|\cdot\|$. If $J=[t_0,t_1]$ is any interval of R, the usual Lebesgue space of square integrable functions from J to R^n will be denoted by $L_2(J,R^n)$.

Let $h>0$ be a given real number and let Q denote the Banach Space of continuous $R^n\times R^m$ - valued functions defined on J with the norm $\|(x,u)\|=\|x\|+\|u\|$, where $\|x\|=\sup\{|x(t)|,\ t\in[t_0,t_1]\}$ and $\|u\|=\sup\{|u(t)|,\ t\in[t_0,t_1]\}$.

That is, $Q=C([-h,0],R^n)\times C([-h,0],R^m)$ where $C=C([-h,0],R^n)$ is the Banach Space of continuous R^n - valued functions defined on J with the supremum norm.

Consider the linear neutral systems with time-varying multiple delays of the form

$$\frac{d}{dt}D(t,x_t)=\sum_{i=0}^{p}A_i(t)x(t-h_i)+\sum_{i=0}^{p}B_i(t)x(t-h_i) \qquad (2.1)$$
$$x(t)=\phi(t),\ t\in[-h,0]$$

and the nonlinear system

$$\frac{d}{dt}D(t,x_t)=\sum_{i=0}^{p}A_i(t)x(t-h_i)+\sum_{i=0}^{p}B_i(t)x(t-h_i)\ +f(t,x(t),x(t-h),u(t),u(t-h) \qquad (2.2)$$
$$x(t)=\phi(t),\ t\in[-h,0]$$

where the operator D is given by

$$D(t,\phi)=\phi(0)-g(t,\phi).$$

In general

$$D(t,x_t)=x(t)-g(t,x_t).$$

Here $x\in R^n$ and u is an m-dimensional vector function with $u\in C_m[-h,t_1]$. The $n\times n$ matrices $A_i(t)$ are continuous and the $n\times m$ matrices $B_i(t)$ are continuous and $\phi(t)$ is a continuous vector function on the interval $[-h,0]$. The n-vector functions f and g are respectively continuous and absolutely continuous.

For a function $x:[-h,t_1]\to R^n$, and $t\in[t_0,t_1]$ we use the symbol x_t to denote the function on $[-h,0]$ defined by $x_t(s)=x(t+s)$ for $s\in[-h,0]$. Similarly, for a function $u:[-h,t_1]\to R^m$, and $t\in[t_0,t_1]$ we use the symbol u_t to denote the function on (t_0,ϕ) defined by $u_t(s)=u(t+s)$ for $s\in[-h,0]$.

The following definitions of complete state and relative controllability of system (2.1) or (2.2) are usual.

***Definition* 2.1**

The set $z(t)=\{x(t),x_t,u_t\}$ is said to be the complete state of the system (2.1) at time t.

***Definition* 2.2**

The system (2.1) or (2.2) *is said to be relatively controllable on* J *if, for every initial complete state* $z(0)$ and $x_1\in R^n$, *there exists a control function* $u(t)$ *defined on* $[t_0,t_1]$ such that the solution of the system (2.1) *or* (2.2) satisfies $x(t_1)=x_1$.

A nonantonomous linear homogeneous neutral differential equation is defined to be

$$\frac{d}{dt}D(t,x_t)=L(t,x_t). \qquad (2.3)$$

A function x is said to be a solution of (2.3) if there exist $t_0\in R, a>0$ such that $x\in C([t_0-h,t_0+a],R^n)$, $t\in(t_0,t_0+a)$ and x satisfies (2.3) on $[t_0,t_0+a]$. Given $t_0\in R,\phi\in C$, we say $x(t_0,\phi)$ is a solution of (2.3) with initial value (t_0,ϕ) if there exists an $a>0$ such that $x(t_0,\phi)$ is a solution of (2.3) on $[t_0-h,t_0+a]$ and $x_{t0}(t_0,\phi)=\phi$.

With the condition on (2.1) or (2.2) solutions of (2.1) or (2.2) exist and are unique and depend continuously on initial conditions. Furthermore, if $T(t,t_0):C\to C, t\geq t_0$ is defined by $T(t,t_0)\phi=x_t(t_0,\phi]$, where $x(t_0\phi)$ is the solution of (2.3), then the variation of constants formula yields the existence of an $n\times n$ matrix function $X(t,s)$

defined for $0 \le s \le t+h,\ t \in [0,\infty)$, continuous in s from the right, of bounded variation in $s, X(t,s)=0, t<s\le t+h,$ such that the solution $x(t_0,\phi,u)$ of (2.1) is given by $x_{t_0}(t_0,\phi,u)=\phi$ and

$$x(t,t_0,\phi,u)=T(t,t_0)\phi(0)+\sum_{i=o}^{p}\int_{-h_i}^{0}X(t,s+h_i)A_i(s+h_i)\phi(s)ds\ +\int_{t_0}^{t}X(t,s)\sum_{i=0}^{p}B_i(s)u(s-h_i)ds,\quad t\ge t_0 \quad (2.4)$$

Observe that if we define the $n\times n$ matrix X_0 as

$$X_0(s)=\begin{cases}0 & if \quad s<0\\ I & if \quad s=0\end{cases}$$

we can set $T(t,s)X_0(\theta)=X_t(.,s)\theta \ =X(t+\theta,s)$ so that $X(t,s)=T(t,s)X_0(0)=T(t,s)I=T(t,s)$.
The corresponding solution of the nonlinear system (2.2) is given by

$$x(t,t_0,\phi,u,f)\ =\ T(t,t_0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t,s+h_i)A_i(s+h_i)\phi(s)ds\ +\ \int_{t_0}^{t}X(t,s)\sum_{i=0}^{p}B_i(s)u(s-h_i)ds$$

$$+\int_{t_0}^{t}X(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds \quad (2.5)$$

Equations (2.4) and (2.5) can be written respectively as

$$x(t,t_0,\phi,u,)\ =\ T(t,t_0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t,s+h_i)A_i(s+h_i)\phi(s)ds\ +\sum_{i=0}^{p}\int_{-h_i}^{0}X(t,s+h_i)B_i(s+h_i)u_{t_0}(s)ds$$

$$+\sum_{i=0}^{p}\int_{0}^{t-h_i}X(t,s+h_i)B_i(s+h_i)u(s)ds \quad (2.6)$$

and

$$x(t,t_0,\phi,u,f)\ =\ T(t,t_0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t,s+h_i)A_i(s+h_i)\phi(s)ds\ +\sum_{i=0}^{p}\int_{-h_i}^{0}X(t,s+h_i)B_i(s+h_i)u_{t_0}(s)ds$$

$$+\sum_{i=0}^{p}\int_{0}^{t-h_i}X(t,s+h_i)B_i(s+h_i)u(s)ds+\int_{t_0}^{t}X(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds \quad (2.7)$$

Define

$$p(t)=T(t,t_0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t,s+h_i)A_i(s+h_i)\phi(s)ds$$

$$q(t)=\sum_{i=0}^{p}\int_{-h_i}^{0}X(t,s+h_i)B_i(s+h_i)u_{t_0}(s)ds$$

$$Y(t,s)=\sum_{i=0}^{p}X(t,s+h_i)B_i(t+h_i)$$

and the controllability matrix

$$W(t_0,t)=\int_{t_0}^{t}Y(t,s)Y^T(t,s)ds$$

where T denotes the matrix transpose.
Then equations (2.6) and (2.7) become

$$x(t)=\ p(t)+q(t)+\int_{t_0}^{t}Y(t,s)u(s)ds \quad (2.8)$$

and $$x(t)=\ p(t)+q(t)+\int_{t_0}^{t}Y(t,s)u(s)ds+\int_{t_0}^{t}x(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds \quad (2.9)$$

It is easy to prove as in [3], that the system (2.1), is relatively controllable on J if and only if W is nonsingular. It is clear that x_1 can be obtained if there exist continuous x and u such that

$$u(t)=Y^T(t_1,t)W^{-1}(t_0,t_1)[x_1-p(t_1)-q(t_1)-\int_{t_0}^{t_1}X(t_1,s)f(s,x(s),x(s-h),u(s),u(s-h))ds] \quad (2.10)$$

and

$$x(t)=\ p(t)+q(t)+\int_{t_0}^{t}Y(t,s)u(s)ds+\int_{t_0}^{t}Y(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds \quad (2.11)$$

We now find the conditions for the existence of such x and u. If $\alpha_i \in L_2(J),\, i = 1,2,..., p,$ the $\|\alpha_i\|$ is the L_2 norm of $\alpha_i(s)$. That is, $\|\alpha_i\| = \int_{t_0}^{t} |\alpha_i| ds$. We introduce as in [1] the following notations:

$$k_1 = \max\{\|X(t,s)\| : t_0 \le s \le t \le t_1\},$$
$$k_2 = \max_{t_0 < s < t_1}\{Y(t_1,s)t_1, 1\},$$
$$a_i = 3k_2 \max_{t_0 < t < t_1}\{\|Y^T(t_1,t)\|\|W^{-1}(t_0,t_1)\|\|X(t_1,t)\|\|\alpha_i\|\},$$
$$b_i = 3k_1\|\alpha_i\|,$$
$$c_i = \max\{a_i, b_i\},$$
$$d_i = 3k_1, \max_{t_0 < t < t_1} \|Y^T(t_1,t)\|\|W^{-1}(t_0,t_1)\|[|x_1| + |p(t_1)| + |q(t_1)|]$$
$$d_2 = 3[|p(t_1)| + |q(t_1)|],$$
$$d = \max\{d_1, d_2\}.$$

3.0 Main result

***Theorem* 3.1**

Let measurable functions $\phi_i : R^{n+m} \to R^+ and. L_2$ *functions* $\alpha_{i:}\ J \to R^+, i = 1,2....., p$ *be such that*

$$|f(t,x,x_{t,}u,u_t)| \le \sum_{i=1}^{p} \alpha_i(t)\phi_i(x,u)$$

for every $(t,x,x_{t,}u,u_t) \in J \times R^{n+m}$.

Then the relative controllability of (2.1) *implies the relative controllability of* (2.2) if

$$\lim_{r\to\infty}\sup(r - \sum_{i=1}^{p} c_i\, Sup\{\phi_i(x,u) : \|(x,u)\| \le r\} = \infty \quad . \qquad (3.1)$$

Proof

Define $T : Q \to Q$ by $T(x,u) = (y,v)$,

where

$$v(t) = y^T(t_1,t)W^{-1}(t_0,t_1)\,[x_1 - p(t_1) - q(t_1) - \int_{t_0}^{t_1} X(t_1,s) f(s,x(s),x(s-h),u(s),u(s-h))ds] \qquad (3.2)$$

and

$$y(t) = p(t) + q(t) + \int_{t_0}^{t} Y(t,s)v(s)ds + \int_{t_0}^{t} X(t,s) f(s,x(s),x(s-h),u(s),u(s-h))ds \qquad (3.3)$$

By our assumptions, the operator T is continuous. Clearly the solution u and x to (2.10) and (2.11) are fixed points of T. We shall prove the existence of such fixed points by using the Schauder fixed-point theorem.

Let $\phi_i(r) = Sup\{\phi_i(x,u) : \|(x,u)\| \le r\}$. Since (3.1) holds, there exists $r_0 > 0$ such that $\sum_{i=1}^{p} c_i\psi_i(r_0) + d \le r_0$.

Now let $Q_{r_0} = \{(x,u) \in Q : \|(x,u)\| \le r_0\}$.

If $(x,u) \in Q_{r_0}$ then from (3.2) and (3.3) we have

$$\|v\| \le \|Y^T(t_1,t)\|\|W^{-1}(t_{0,}t_1)\|[|x_1| + |p(t_1)| + |q(t_1)| \quad + \int_{t_0}^{t_1} \|X(t_1,s)\| \sum_{i=1}^{p} \alpha_i(s)\phi_i(x(s),u(s))ds]$$

$$\le \frac{d_1}{3k_2} + \frac{1}{3k_2}\sum_{i=1}^{p} c_i\psi_i(r_0) \le \frac{1}{3k_2}(d + \sum_{i=1}^{p} c_i\psi_i(r_0)) \le \frac{r_0}{3k_2} \le \frac{r_0}{3}$$

and

$$\|y\| \le |p(t)| + |q(t)| + \int_{t_0}^{t} \|Y(t,s)\|\|v\|ds + \int_{t_0}^{t} \|X(t,s)\| \sum_{i=1}^{p} \alpha_i\phi_i(x(s),u(s))ds$$

$$\le \frac{d}{3} + k_2\|v\| + k_1\sum_{i=1}^{p} \|\alpha_i\|\psi_i(r_0) \le \frac{d}{3} + k_2\|v\| + \frac{1}{3}\sum_{i=1}^{p} c_i\psi_i(r_0) \le \frac{1}{3}(d + \sum_{i=1}^{p} c_i\psi_i(r_0)) + k_2\|v\|$$

$\leq \frac{r_0}{3} + \frac{r_0}{3} = \frac{2r_0}{3}$.

Hence T maps Q_{r_0} into itself. Since all the functions involved in the definition of the operator T are continuous, it follows that T is continuous. By the Ascoli-Arzela theorem, $T(Q_{r_0})$ is compact in Q. Since Q_{r_0} is closed, bounded and convex, the Schauder fixed- point theorem guarantees that T has a fixed point $(x,u) \in Q_{r_0}$ such that $T(x,u) = (x,u)$. It follows that, for $(x,u) \equiv (y,v)$, we have

$$x(t) = p(t) + q(t) + \int_{t_0}^{t} Y(t,s)u(s)ds + \int_{t_0}^{t} X(t,s) f(s, x(s), x(s-h), u(s), u(s-h))ds .$$

Thus the solutions of (2.10) and (2.11) exist. Hence the system is relatively controllable on J .

4.0 Conclusion

Using Schauder's fixed point theorem sufficient conditions for the relative controllability of nonlinear neutral functional differential systems with time varying multiple delays in the state and control have been derived. The results extend those of [12] to neutral systems.

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