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Euclidean null controllability of linear systems with delays in state and control

Davies Iyai
Department of Mathematics and Computer Science,
Rivers State University of Science and Technology,
Port Harcourt, Nigeria.
email: davsdone@yahoo.com

Abstract

Sufficient conditions are developed for the Euclidean controllability of linear systems with delay in state and in control. Namely, if the uncontrolled system is uniformly asymptotically stable and the control equation proper, then the control system is Euclidean null controllable.

***Keywords*:** Admissible controls, Controllability, Delayed systems, Linear systems, Null controllability.

1.0 Introduction

The study of controllability for linear systems, first carried out in details by R.E. Kalman and his co-workers in the 1960's have attracted lots of literature in modern research because of its wide application to many fields. In recent years, the problem of controllability of systems with state delays has been considered (See [8, 9]). Kloch [8], considered the control system described by the equation

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t)u(t-i) + Cu(t) \qquad (1.1)$$

and gave an algebraic necessary and sufficient condition for the normality of the system (1.1). Controllability conditions for systems with delays in the control are given in [1, 7, and 11]. In particular, [11] has given controllability conditions for a class of linear systems described by the equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)u(t-h) \qquad (1.2)$$

using rank criterion and found an explicit expression for transferring a given state to any desired state using minimum control energy. It is known therein that, systems with delayed control are natural models for the study of some economic, biological and physiological systems as well as electromagnetic systems composed of subsystems interconnected by hydraulic and various other linkages.

It will be interesting to forge ahead to study systems with delay in both state and control variables. Mathematical models for such systems play important role in every field of science and engineering where causes do not produce their effects immediately but, with some time delays. Systems with delay in both state and control were studied in [10]. Here, criteria for controllability were formulated for the differential equation

$$\dot{x}(t) = \sum_{i=0}^{p} A_i(t)\, x(t-h_i) + \sum_{i=0}^{p} B_i(t) u(t-h_i) \qquad (1.3)$$

with the aid of some sequence of matrices resulting from determining equation, where the rank of these sequence determines the controllability property for the system (1.3).

In this research we shall examine the Euclidean null controllability of linear systems with delay in state and control variable defined by

$$\dot{x}(t) = L(t, x_t) + B(t)u(t-h) \qquad (1.4)$$

That is, steering, the solution of our system from an initial state to the origin in finite time using admissible controls. Our aim is to incorporate the idea of rank criterion from the determining equation ([8]), and extend this by exploiting the property of the reachable set and the stability of the uncontrolled system to give sufficient criteria for the Euclidean null controllability of (1.4).

2.0 Basic notations and preliminaries

Suppose $h > 0$ is a given number, let E denote the real line. For a positive integer n, E^n is a real n-dimensional Euclidean space with norm $|\cdot|$.

In this paper, let $W_2^{(1)}$ denote the sobolev space $W_2^{(1)}([-h,0], E^n)$ consisting of all absolute continuous functions $\phi:[-h,0] \to E^n$, whose derivatives are square integrable. If $x:[t_0 - h, t_1] \to E^n$, $t \in [t_0, t_1]$, we use the symbol x_t to denote the function on $[-h,0]$ defined by $x_t(s) = x(t+s)$ for $s \in [-h,0]$. The control u is a measurable m-vector valued function with values in $C^m = \{u \in E^m : |u_j| \le 1, \ j = 1,2,........m\}$.

Consider our system of interest

$$\dot{x}(t) = L(t, x_t) + B(t)u(t-h) \tag{2.1}$$

where

$$L(t,\phi) = \sum_{k=0}^{N} A_k(t)\phi(-h_k) \tag{2.2}$$

satisfied almost everywhere on $[t_0, t_1]$. Where $L(t,\phi)$ is continuous in t, linear in ϕ. Each A_k is a continuous $n \times n$ matrix function for, $0 \le h_k < h$. $B(t)$ is an $n \times n$ matrix function.

The solution of system (2.1) satisfying $x(t) = \phi(t)$ for $t > t_0 + h$ is given as

$$x(t, t_0, \phi, u) = X(t, t_0)\phi(t_0) + \int_{t_0}^{t} X(t,s)B(s)u(s+h)ds \tag{2.3}$$

where $X(t,s)$ satisfies the equation

$$\frac{\partial X(t,s)}{\partial t} = L(t, X_t(\cdot, s)), \ t > s$$

almost everywhere

$$X(t,s) = \begin{cases} 0, & s-h < t < s \\ I, & t = s \ \ (I \equiv Identity\, matrix) \end{cases}$$

$X(t,s)$ is the fundamental matrix solution of the system

$$\dot{x}(t) = L(t, x_t) \tag{2.4}$$

We obtain a more convenient form of the solution (2.1) by expressing (2.3) in the form

$$x(t, t_0, \phi, u) = x(t) = X(t, t_0)\left[x(t_0) + \int_{t_0-h}^{t_0} X(t_0, s+h)B(s+h)u_{t_0}(s)ds\right] + \int_{t_0}^{t-h} X(t, s+h)B(s+h)u(s)ds \tag{2.5}$$

for simplicity of notation, let $Z(s) = X(t, s+h)B(z+h)$ and define the reachable set of system (2.1) at time t by

$$R(t) = \left\{\int_{t_0}^{t-h} Z(s)u(s)ds : u \in C^m\right\} \tag{2.6}$$

We now give some basic definitions upon which our study hinges.

***Definition* 2.1**

The system (2.1) *is said to be proper in* E^n *on an interval* $[t_0, t_1]$ if $c^{\bullet}Z(s) = 0$, almost *everywhere* $s \in [t_0, t_1]$, $c \in E^n$ *implies* $c = 0$. *If system* (2.1) *is proper on* $[t_0, t_0 + \delta]$ *for each* $\delta > 0$ *we* say that system (2.1) *is proper at time* t_0. *If system* (2.1) *is proper on each interval* $[t_0, t_1]$, $t_1 > t_0 \ge 0$ we *say that the system is proper in* E^n.

Definition 2.2

The domain ψ of Euclidean null controllability is the set of initial functions in $W_2^{(1)}$ which can be steered to the origin $0 \in E^n$ in finite time, using admissible controls.

***Definition* 2.3**

The trivial solution of system (2.4) *with initial function $x(t) = \phi(t)$, $t \in [-h,0]$ is called stable at t_0* if $t_0 \geq 0$ *and*

(i) *There exists* a $b = (t_0) > 0$ *such that if* $\|\phi\| \leq b$. *Then the solution* $x(t_0, \phi)$ *of* (2.4) *exists for* $t \geq t_0$

(ii) *For each $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \phi)$* such that if $\|\phi\| \leq \delta$, *then the solution* $x(t, t_0, \phi)$ *of* (2.4) *satisfies* $\|x_t(t, t_0, \phi)\| \leq \varepsilon$ *for all* $t \geq 0$. *The trivial solution is called stable if it is stable for each* $t_0 \geq 0$ *.It is called uniformly stable if it is stable and the* δ *above does not depend on* t_0. *I is uniformly asymptotically stable if it is uniformly stable and for each* $\eta > 0$ *and every* $t_0 > 0$ *there exists* $T(\eta)$ *independent of* t_0 *and there exists* $H_0 > 0$ *independent of* η, t_0, *such that* $\|\phi\| \leq H_0$ *implies* $\|x_t(t, t_0, \phi)\| < \eta$ for $t \geq t_0 + T(\eta)$ (See [5]) .

***Definition* 2.4**

System (2.1) *is Euclidean null controllability if for each* $\phi \in W_2^{(1)}$, $x_1 \in E^n$ *there exists a* $t_1 > 0$ *and an admissible control* u *such that the solution* $x(t;\, \phi, u)$ of (2.1) s*atisfies* $x_0(\phi, u) = \phi$ *and* $x(t_1;\, \phi, u) = 0$

***Proposition* 2.1**

The reachable set $R(t)$ *is symmetric convex and closed. Also* $0 \in \operatorname{int} R(t)$ *for each* $0 \leq s$.

The proof is similar to corresponding results for linear control systems of various types, for details (See [2]).
Following [4, 12] we introduce the determining equations for the system

$$\dot{x}(t) = \sum_{k=0}^{N} A_k x(t - h_k) + B(t)u(t - h) \qquad (2.7)$$

where $0 < t_0 < t_1$ and A_0, A_1 and B are constant matrices, as

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s - h), \quad k = 1, 2, \ldots \quad s \in (-\infty, \infty)$$

$$Q_0(s) = \begin{cases} c, & s = 0 \\ 0, & s \neq 0 \end{cases} \qquad (2.8)$$

Define

$$\hat{Q}_n(t_1) = \{Q_0(s), Q_1(s), \ldots\ldots, Q_{n-1}(s);\ s \in [t_0, t_1]\} \qquad (2.9)$$

***Proposition* 2.2**

The control system (2.7) is proper in E^n on the interval $[t_0, t_1]$ if and only if rank $\hat{Q}(t_1) = n$.
For the proof (See [12]).

***Theorem* 2.1**

System (2.1) *is proper on* $[t_0, t_1]$, $t_1 > t_0$ *if and only if* $0 \in \operatorname{int} R(t_1)$.

Proof

It is known in [6] that $R(t_1)$ is a closed and convex subset of E^n. Therefore, a point y_1 on the boundary of $R(t_1)$ implies there is a support plane π of $R(t_1)$ through y_1, that is $\eta^{\bullet}(y - y_1) \leq 0$ for each $y \in R(t_1)$ where $\eta \neq 0$ is an outward normal π. If u_1 is the control corresponding to y_1 we have

$$\eta^{\bullet} \int_{t_0}^{t_1 - h} Z(s)u(s)ds \leq \eta^{\bullet} \int_{t_0}^{t_1 - h} Z(s)u_1(s)ds$$

for each $u \in C^m$. Since C^m is a unit cube this last inequality holds for each $u \in C^m$ if and only if

$\eta^{\bullet}\int_{t_0}^{t_1-h} Z(s)u(s)ds \le \int_{t_0}^{t_1-h} \left|\eta^{\bullet} Z(s)u(s)\right| ds = y_1 = \int_{t_0}^{t_1-h} \left|\eta^{\bullet} Z(s)\right| ds$ and $u_1(s) = \operatorname{sgn} \eta^{\bullet} Z(s)$ since we always have $0 \in R(t_1)$. If 0 were not in the interior of $R(t_1)$ then 0 is on the boundary. Hence, the preceding argument implies $0 = \int_{t_0}^{t_1-h} \left|\eta^{\bullet} Z(s) ds\right|$. So that $\eta^{\bullet} Z(s) = 0$ almost everywhere $s \in [t_0, t_1]$. This by definition of proper implies that the system is not proper since $\eta^{\bullet} \ne 0$. This completes the proof.

***Theorem* 2.2**

System (2.1) *is proper in* E^n *, then* ψ *, the domain of Euclidean null- controllability contains zero in its interior.*

Proof

Since $x(t) = 0$ is a solution of (2.1) with $u = 0$ implies that $0 \in \psi$. Assume that system (2.1) is proper, then 0 is in the interior of $R(t)$ for each t . If 0 were not in the interior of ψ then there is a sequence $\{x_m\}_1^{\infty} \subseteq C$ such that $x_m \to 0$ as $m \to \infty$ and no x_m is in $\psi(\textit{so } x_m \ne 0)$ from the variation of parameter, every solution of (2.1) with $u = 0$ satisfies.

$$0 \ne x(t_1, t_0, x_m, 0) = X(t_1, t_0) x_m(t_0) + \int_{t_0}^{t_1-h} Z(s)u(s)ds$$

for any $t_1 \ge 0$ and any $u \in C^m$. Hence, $r_m \overset{def}{=} x(t_1, t_0) x_m(t_0)$ is not in $R(t_1)$ for any $t_1 \ge 0$. Therefore the sequence $\{r_m\}_1^{\infty} \subseteq E^n$, $r_m \in R(t_1) r_m \ne 0$ is such that $r_m \to 0$ as $m \to 0$. Therefore, 0 is not in the interior of $R(t_1)$ for any t_1 , a contradiction. Hence, $0 \in \operatorname{int} \psi$.

3.0 Controllability results

***Lemma* 3.1**

The solution $x = 0$ *of system* (2.1) *with* $u = 0$ *is uniformly stable if and only if there exists a constant* β_1 *, such that for all* $s \in E^n$, $\left|X(t,s)\right| \le \beta_1$, $t \ge s$

Lemma 3.2

The solution $x = 0$ of system (2.1) with $u = 0$ *is uniformly asymptotically stable if and only if there exist constants* $\beta_2 > 0$, $\alpha > 0$ such that $\left\| x_t(\phi) \right\| \le \beta_2 \left\| \phi \right\| e^{-\alpha t}$, $t \ge 0$ (see [5]).

3.1 Main result

***Theorem* 3.1:** *In system* (2.1) *assuming that*

(i) (2.1) *is proper in* E^n

(ii) (2.1) *with* $u = 0$ is *uniformly asymptotically stable then system (2.1) is Euclidean null controllable with constraints.*

Proof

By (i) assume system (2.1) is proper in E^n, then by theorem 2.2 the domain ψ of null controllability contains an open ball S of finite radius around the zero function ϕ_0. By (ii), every solution of (2.1) (with $u = 0$) let an initial function $\phi_1 \in W_2^{(1)}$ be given. Using the null control $u(t) = 0$ the solution (2.1) (with $u = 0$) satisfies $x_t(\phi_1, 0) \to \phi_0 \equiv 0$ as $t \to \infty$. Hence, at some $t_1 < \infty$, $x_{t_1}(\phi_1, 0) \in S$. Therefore, using t_1 for some finite time $x_{t_1}(\phi, 0) \subseteq \psi$. But then $x_{t_1}(\phi_1, 0)$ can be steered to $0 \in E^n$ in finite time. Hence (2.1) is Euclidean null controllable.

***Example* 3.1**

Consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B(t)u(t-h) \qquad (3.1)$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -q \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and $$b > q, \quad h > 0 \qquad (3.2)$$

The characteristic roots of the homogeneous equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) \qquad (3.3)$$

is $$\lambda^2 + b\lambda + q\lambda\, e^{-\lambda h} + k = 0 \qquad (3.4)$$

and every root of (3.4) by [3] has negative real part if $b > q$ and by [5] it is uniformly asymptotically stable.

To show properness of (3.1) using the determining equations for the interval $[t_0, t_1], \ 0 < t_0 < t_1$.

$$Q_0(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad Q_0(s) = 0, \quad s \neq 0$$

$$Q_1(0) = A_0 B = \begin{bmatrix} -1 \\ b \end{bmatrix}, \quad Q_1(h) = A_1 B = \begin{bmatrix} 0 \\ q \end{bmatrix}$$

$$Q_1(s) = 0, \quad s \neq 0, \quad s \neq h$$

$$\hat{Q}_1(t) = \begin{bmatrix} 0 & -1 \\ -1 & b \end{bmatrix}, \quad h > t > 0$$

$$\hat{Q}_1(2h) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & b & q \end{bmatrix}$$

Since rank $\hat{Q}_2(t) = 2$ for each $t > 0$ the system is proper on each $[t_0, t_1], \ 0 < t_0 < t_1$ on E^n. Hence we conclude that system (3.1) is Euclidean null controllable.

4.0 Conclusion

Sufficient conditions for the Euclidean controllability of linear systems with delay in state and control have been derived. These conditions were given with respect to the stability of the uncontrolled base system and the properness of the linear controllable base system. It was shown that, a linear system with delay in state and control variables can be reinstated to the origin in finite time if the uniform asymptotic stability of the free system is guaranteed and the rank criterion of the determining equation for the controllable system satisfied. Our result compliment and extends other known results on the subject

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