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Necessity and sufficiency conditions for the absolute null controllability for Linear delay perturbations

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Abstract

We are inspired by the works of Chukwu [1], Eke [2], Schinterdorf and Barmish [4] to unveil necessary and sufficient conditions for the absolute null controllability of a linear delay perturbed system with zero in the domain of null controllability.

1.0 Introduction

Perturbations occur frequently in physical systems, amplification or saturation. Even linear systems with delay introduce perturbed feedback as in relay control. The present work is concerned with addressing this problem. A delay in a state or control variable affects the evolution of the system in an indirect manner. An example of such delay system is given by:

$$\dot{x}(t) = L(t, x_t) + \int_{-h}^{o} d_s H(t,s)u(t+s) + f(t,x,u) \qquad (1.1)$$

and it is the focus of our attention. While controllability of a system involves steering a solution of a system from an initial point to a terminal point using an appropriate control, null controllability is concerned with re-instating a system to equilibrium state or controlling the system to the origin.

Considerable number of researches have been carried out on these areas (for example [3], [4]). For the ordinary control system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (1.2)$$

and the linear perturbation

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) + f(t, x, u) \qquad (1.3)$$

the controllability abound in the literature. In [1], Chukwu showed that if the homogeneous part of the system (1.3) is proper, then the linear perturbation is null controllable provided the perturbation function f satisfies certain growth and continuity conditions. Quite recently a few researches have been reported on systems with delays in the control variable. From these, two types of controllability were defined in appreciation of the effect of delay on the control. They are Relative and Absolute Controllability. In [3], Nse resolved the former while the latter is the focus of this work.

Owing to the obvious difficulty in handling the many lags in both the state and control variables, only a few studies have been carried out in the absolute controllability of system (1.1) with perturbation. This is the thrust on which this investigation is based.

2.0 Notations and definitions

Let m and n be positive integers, E, the real n- tuples with the Euclidean norm denoted by $\|.\|$ J is an internal of E, the usual Lebesgue space of square integrable functions from J to E^n denoted by $L_2(J,E^n)$. $M_{n,m}$ will be used for the collection of all nxm matrices with a suitable norm.

Let $h > 0$ be given, for functions $x:[t_o-h,t,]\rightarrow E^n$, $t \in [to, t_1]$ we use x_t to denote the function on $[-h,0]$ defined by $x(t)$ x(t + s) for s$\in$[-h,0]. $C = C([-h, 0], E^n)$ is the space of continuous functions mapping the interval $[-h,0]$ into E^n.

Similarly, for functions u:$[-h,0]\rightarrow E^m$, $t \in [to,t_1]$. We use u_t to denote the function on $[-h,0]$ defined by $u_t(s) = u(t + s)$ for s$\in[-h,0]$. Given the system as in (1.1), we let

$$L(t,x_t) = \sum_{k=0}^{n} A_k(t)x(t-w_k + \int_{-h}^{0} A(t,\theta)x(t+\theta) \qquad (2.1)$$

satisfied almost everywhere on $[t_0,t_1]$ where the integral is in the Lebesgue-Stieljes sense with respect to s. $x(t) \varepsilon$ C, $u \varepsilon L_2([to,t_1],E^n)$. $L(t,\Phi)$ is continuous in t, linear in Φ and $H(t,s)$ is an n x m matrix valued function measurable in (t,s) . We shall assume that $H(t,s)$ is of bounded variation in s on $[-h,0]$ for each $t \varepsilon [t_o,t_1]$ with $var_{[-h,0]} \leq m(t)$ where $m(t)\in L_1([t_o,t_1],E)$ and H(t,s) are absolutely continuous in s on [-h,0]. $A\in L_1([t_o,t_1],M_{m,n})$. In this work ,the control of interest are from $B = L_2([t_o,t_1],E^n)$ where U is closed and a bounded subset of B with zero in the interior relative to B. If X and Y are linear spaces and $T:X \rightarrow Y$ is a mapping , we shall use symbols $D(t)$,$R(t)$, and $N(t)$ to denote the Domain, Range and Null spaces respectively of T while it is assumed $f(t,x,u)$ behaves as in [1]. We now state some definitions to ensure understanding of our main result.

***Definition* 2.1** (Complete state)

The complete state of system (1.1) *at time t is given by*

$$z(t) = \{x(t),u(t)\} \qquad (2.2)$$

***Definition* 2.2**

The solution of the base system (1.1) *is of the form*

$$x(t,t_o,\Phi,u) = X(t,t_o,\Phi) \int_{t_0}^{t} X(t,1)\,[\, \int_{-h}^{0} d_1 H(1,S) u(1+s) \qquad (2.3)$$

where

$$X_t(\bullet\ ,s)\theta = X(t+\theta,s), \quad -h \leq \theta \leq 0 \qquad (2.4)$$

and $X(t,s)$ *is the fundamental matrix solution of* $x(t) = L(t,x_t)$ *satisfying*

$$\frac{\partial X(t,s)}{\partial t} = L(t,x_t(\cdot,s))\ ,\ t \geq s \qquad (2.5)$$

almost everywhere in (t,s) *and*

$$X(t,s)\begin{cases} 0 & \text{when } s - h \leq t < s \\ I & \text{when } t = s \end{cases}$$

I *is the identity matrix*.

***Definition* 2.3** (Absolute Null Controllability)

The base system (1.1) *is said to be absolutely controllable at* $t = t_1$ *if for any initial complete state* $z(t_o) = \{x(t_o),u(t_o)\}$ *on* $[t_o-h,t_o]$, *there exists an admissible control* $u(t) \in B$ *defined on* $[to,t1-h]$ *such that the response of the system x(t) satisfies* $x(t1) = 0$ *using the control*;

$$u(t) = \begin{cases} (t) & \text{on } [t_0,t_1\text{-}h] \\ 0 & \text{on } [t_1\text{-}h,t_1\text{-}h] \end{cases} \qquad (2.6)$$

***Definition* 2.4** (Domain of null controllability)

The domain of null controllability of system (1.1) *is the set of all initial functions* $\varphi \in$ C for *which the solution of* (1.1) *with* $x_o = \Phi$ *satisfies* $x(t_1)$ *for* $t_1 > t_o$, $u \varepsilon U$

The above conditions on $L(t,\Phi)$ and H ensure the existence of a unique absolutely continuous solution $x(t)$ of (1.1) with initial complete state $z(t_o)$

3.0 Main eesults

Here we give a theorem that summarizes the major results on the absolute null controllability of system (1.1)

***Theorem* 3.1**

Consider system (1.1) *given by*

$$\dot{x}(t) = L(t,x_t) + \int_{-h}^{0} d_s H(t,s)u(t+s) \qquad (3.1)$$

with the basic assumptions. The system (3.1) *with the control* $u_o(t)$ on $[t_o-h,t_o]$ *is absolutely null controllable at* $t = t_1$ *if and only if*

$$y(z(t_o)) = -\Phi(t_o) - \int_{-h}^{0} ds\left[\int_{t_0+s}^{t_0} X(t_1, l-s)H(l-s,s)u_{t_0}dl\right]$$

belongs to the range space of the null controllability gramian $\Gamma(t_o,t_1) = \int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,l\text{-}s)d\overline{H}(l\text{-}s,s)][\int_{-h}^{0} X(t_1,l\text{-}s)dH(l\text{-}s,s)]^T dl$

Proof

(i) **Sufficiency**:

Let $y(z(t_o)) \varepsilon R(\Gamma(t_o,t_1))$, then for $z \in E^n$, we have $y(z(t_o)) = \Gamma(t_o,t_1)z_o$. Choose

$$\begin{cases} \int_{-h}^{0}\left[X(t_1,l-s)d\vec{H}(l-s.s)^T z_0\right], \text{ for } l \in [t_0,t_1-h] \\ 0, \text{ for } l \in [t_0,t_1-h,t_1] \end{cases} \qquad (3.2)$$

then substituting (3.2) into the variation of parameter equation for (3.1), we obtain

$x(t_1,t_o,\Phi,u) = X(t_1,t_o)[\Phi(to)+\int_{-h}^{0} dH \int_{t_{0+s}}^{t_1} X(t_1,l\text{-}s)\overline{H}(l\text{-}s)u_o dl+\int_{tO}^{t1}\int_{-h}^{0} X(t_1,l\text{-}s)dH(l\text{-}s,s)u(l)dl$

$= X(t_1,t_o)[-y(z(t_o))]+X(t_1,t_o)\Gamma(t_1,t_o) = -X(t_1,t_o)\Gamma(t_o,t_1)z_o+X(t_1,t_o)\Gamma(t_1,t_o)z_o = 0$

establishing absolute null controllability.

(ii) **Necessity**

Suppose for a contradiction that $y(z(t_o)) \notin R(\Gamma(t_o,t_1))$ then there exists $z_1,z_2 \varepsilon E^n$ such that $y(z(t_o)) = z_1 + z_2$; $z_2 \neq 0$ where $z_1 \in R(\Gamma(t_o,t_1))$, $z_2 \in N(\Gamma(t_o,t_1))$ Thus $\langle z_2\Gamma(t_o,t_1),z_2\rangle = \int_{to}^{t1-h}\|\Gamma(t_o,t_1)z_2\|^2 dl = \int_{to}^{t1-h}\| X(t_o,l\text{-}s)d\overline{H}(l\text{-}s,s)^T z_2\|^2 dl$. Since the integrand is non-negative, we obtain

$[\int_{-h}^{0} X(t_o,l\text{-}s)d\,d\overline{H}(l\text{-}s,s)]^T z_2 = 0$, $l \in [t_o,t_1\text{-}h]$. By hypothesis, however, $(X(to),x_{to},u_{to})$ can be brought to the origin by some control on $[t_o,t_1]$. That is

$$\int_{to}^{t1-h}[\int_{-h}^{0} X(t_o,l\text{-}s)d\overline{H}(l\text{-}s,s)]u(l)dl = 0 \qquad (3.4)$$

$$\Rightarrow \qquad z_2^T[\int_{-h}^{0} X(t_o,l\text{-}s)\,d\overline{H}(l\text{-}s,s)] = 0 \qquad (3.5)$$

By combining (3.3) and (3.5), we obtain a contradiction: $\|z_2\|^2 = 0$ since $z_2 \neq 0$.

Hence $y(z(t_o)) \in R(\Gamma(t_o,t_1))$. This completes the proof.

4.0 Conclusion

This work settles the question of absolute null controllability of the system under study in the affirmative .It is seen that the equality of the Euclidean space E^n and the range space of the null controllability gramian is a necessary and sufficient condition for the absolute null controllability of system (1.1). It has also been shown that under certain smoothness conditions of the perturbation function, controllability is also guaranteed. This work however extends known results on the controllability of ordinary control systems to the controllability of systems with delays.

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