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# Stability of discrete control systems

***Celestin. A. Nse***
**Department of Mathematics and Computer Sciences**
**Federal University of Technology, Owerri, Nigeria**.

**Abstract**

**Consider the discrete control system**

$$x(t+1) = Ax(t) + Bu(t),\ y(t) = Hu[t] \qquad \textbf{(*)}$$

**for** $t \in T = \{0,1,2,\cdots\}$

**We implore the notion of asymptotic controllability to show that, a system which can be stabilized by an arbitrary feedback** $u = g(x)$ **can also be stabilized by a linear feedback** $u = D(x)$**.**

## 1.0 Introduction

Stability is of importance for the synthesis of feedback controls. A control is called a feedback if it is described as a function of the state variable x, that is y=g(x). La Salle [4] pointed out that an important condition for linear feedback stability is controllability. No wonder enthusiasts in controllability have picked interest in the subject of stability as it applies to controllability.(See [2],[3],[4]). In his contribution on the subject, Eke [3] provided a shorter and rather easier proof of the stability conditions for systems of the form

$$x(t) = Ax(t) + Bu(t),\ x(0) = 0$$
$$y(t) = Hu(t) \qquad (1.1)$$

His paper however touched slightly the fringes of functional analysis and of course such basic linear algebra that could be shown to any serious person in need of the stability of control systems. Be that as it may, the object of this paper is to provide a rigorous foundation for the theory of stability of discrete control processes and motivate a closer examination of continuous autonomous systems as a possible objective in control design. Classical optimal control theory provides several examples of systems that exhibit convergence to the equilibrium in finite time. A well known example is the double integrator with bang bang time optimal feedback control [1].These examples typically involve dynamics that are discontinuous (discrete). Discontinuous dynamics besides making a rigorous analysis difficult may also lead to chattering or excite high frequency dynamics in applications using time-varying feedback controls. However it is well known that the stability analysts of discrete time-varying systems is more complicated than that of continuous systems Therefore with simplicity as well as applications in mind, we attempt here to alleviate this difficulty. We thus gather the tools necessary in the execution of this task.

In (*), A, B, H denote real matrices of dimensions $n \times n,\ n \times m,\ r \times n,$ respectively. Here the functions *u, x, y* are defined on *T* and are vector-valued. The set of admissible controls (or input variables) is defined by Ω and consists of all sequences *u*= {*u*(0),*u*(1),….}.The function x is called the state variable and y the output variable. For every *u* ε Ω, *a* ε $R^n$, the solution of (*) corresponding to u with initial value *x*(0) = a is denoted by $x_u(t,a)$.

***Definition* 1.1**

*The system* (*) *is called controllable for every a, b ε $R^n$, if there exists a u ε Ω, t ε T such that* $x_u(t,a) = b$.

***Definition* 1.2**

*System* (*) *is called null controllable if for every* a ε $R^n$ *there exists u ε Ω and t* ε T *such that* $x_u(t,a) = 0$. *It is called asymptotically controllable if for every a ε $R^n$ ,there exists u ε Ω such that* $x_u(t,a) \to 0$ *as* $t \to \infty$.

***Definition* 1.3**

*The set of eigenvalues of A is called the spectrum of A and is denoted by δ(A). The characteristic polynomial of A is denoted by $\chi_A$ and is defined by*

$$\chi_A(z) = det(zI - A)$$

*An eigenvalue λ of A is called* (*)–***stable*** *if* $|\lambda| < 1$. *Eigenvalues of A which are not (*)–stable are called* ***(*)–unstable***. *The matrix A is called* (*)–*stable if all eigenvalues of A are* (*)–*stable.*

***Definition* 1.4**

*The system* (*) *is called* ***stabilizable*** *if there exists an m* x *n matrix D such that A+BD is* (*)-*stable.*

## 2.0 Controllability and direct stabilizability

We have the following results.

***Theorem* 2.1**

*If S is a non empty set of complex numbers, there exists a matrix D with δ(A+D) cS if and only if every* $\lambda \in \delta(A) \setminus S$ *is controllable.*

**Proof**

Suppose that each $\lambda \in \delta(A) \setminus S$ is controllable, a transformation of state space $\bar{x} = Tx$ transforms (*A*, *B*) into $(\bar{A}, \bar{B}) := (T^{1}AT, T^{-1}B)$ and is easily seen that (A,B)-controllable eigenvalues are also $(\bar{A}, \bar{B})$-controllable. By the canonical decomposition theorem ([4], p 99), we can choose T such that we have the following block-partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where $\bar{A}_{11}$ is a *p* x *p* matrix and $B_1$ is a *p* x *m* matrix such that $(\bar{A}_{11}, \bar{B}_1)$ is controllable. We have δ(A) = δ(A) = δ($A_{11}$) *U*δ($A_{22}$). If λ ε δ($A_{22}$) , then λ is not (*A*, *B*)-controllable and hence λ ε S. Therefore we have δ($A_{22}$) c *S*. Furthermore, since ($A_{11}$, $B_1$) is controllable, it follows that, there exists *D* with δ($A_{11}$ + $B_1D_1$) c *S*.

***Theorem* 2.2**

*The following statements are equivalent.*

*(i)* *system* (*) *is asymptotically controllable.*

*(ii)* *Every unstable eigenvalue of A is controllable.*

*(iii)* (*) *is stabilizable.*

**Proof:**

(i)⇒ (ii) If for λ ε *δ(A)*. With *Re*$\lambda \geq 0$, there exists a row vector $\eta \neq 0$ with $\eta A = \lambda\eta$, $\eta B = 0$ and if $\eta a \neq 0$, then we have $(d/dt)(\eta x_u(t,\alpha)) = \lambda\eta x_u(t,\alpha)$. Hence $\eta x_u(t,\alpha) = e^{\lambda t}\eta \to 0$, ($t \to \infty$) for every u ε Ω. Therefore (*) is asymptotically controllable.

(ii) ⇒ (iii) This is an immediate consequence from Theorem 2.1.

(iii) ⇒ (i) Let D stabilize (*) and let the solution of *x* = (*A*+*BD*) with *x*(0) = *a* be denoted by *x*^(*t*,*a*). Then with *u*(*t*):= *Dx*(*t*,*a*), we have $x_u(t,a)$.= *x*^(*t*,*a*), (*t* > 0) and hence $x_u(t,a) \to 0$, ($t \to \infty$).

**Remark**

It follows from this Theorem in particular that a system which can be stabilized by an arbitrary feedback *u* = *g*(*x*) can also be stabilized by a linear feedback *u* =*D*(*x*).

## 3.0 Conclusion

The notion of asymptotic stability can precisely be formulated within the framework of asymptotic controllability with straightforward uniqueness. This assumption however does not imply any regularity property for the settling time function. Stabilizable results for finite time stability naturally involve finite time scalar differential equalities. The regularity properties of a stabilizable function satisfying such equalities strongly depend on the regularity properties of the settling time function.

This paper thus raises certain questions that are important from the point of view of stability theory. In particular, conditions on the dynamics for the settling time function that lead to a stronger converse result is of interest to the mathematical physicist. As mentioned earlier, a control system under the action of a time optimal feedback controller, yields a close loop system that exhibit finite time convergence. Hence it would be interesting to explore the connections between finite time stability and time optimality and relate the results of this paper to result of the time optimal control problem.

## References


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