*Journal of the Nigerian Association of Mathematical Physics*
*Volume 10 (November 2006)*, **543 - 546**


# Relative controllability of nonlinear neutral Volterra Integrodiferential systems with delays in control

***P. C. Jackreece***
**Department of Mathematics and Computer Science**
**Niger Delta University, Wilberforce Island,**
**Amasoma, Nigeria.**
**e-mail-preboj@yahoo,com**

**Abstract**

**In this paper we established sufficient conditions for the relative controllability of the nonlinear neutral volterra integro-differential systems with distributed delays in the control. The results were established using the Schauder's fixed point theorem which is an extension of known results.**

## 1.0 Introduction.

The study of functional differential equations with various types of delays has attracted the attention of many researchers in recent years, because of its connection to many fields, such as population dynamics, theoretical epidemiology, systems theory, ecology, transmission line theory, etc (see Hale [10], Han et el [9] and Balachandra et el [6]). In Gyori and Wu [8], a simplified model for compartmental systems with pipes is represented by nonlinear neutral volterra integrodifferential systems. A concrete example of compartmental model is the Radiocardiodram where two compartments represent the left and right ventricles of the heart and the pipes between these compartments represents the pulmonary and systematic circulation. Pipes coming from and returning into the same compartment may represent shunts and corollary circulation.

The problem of controllability of neutral systems has been investigated by several authors. Angell [1], Balachandra and Dauer [3], discussed the controllability of nonlinear neutral systems Balachandra [2] established sufficient conditions for the controllability of nonlinear neutral volterra integrodifferential systems. Further Balachandra and Balasubramaniam [5] established conditions for the controllability of nonlinear neutral volterra integrodifferential systems with implicit derivative.

However Balachandra and Anandhi [4], established sufficient condition for the controllability of neutral functional integrodifferential infinite delay systems in banach space using analytic semigroup theory and Nussbaum fixed point theorem. In [14], Xianlang used the fractional power of operators and sadovskii fixed point theorem to study controllability and local controllability of abstract neutral functional differential systems with unbounded delays. In this paper we shall study the relative controllability of nonlinear neutral volterra integrodifferential systems with distributed delays in control by suitably adapting the technique of Do [7]. This is an extension of the work of Balachandra and Dauer [3] were the matrix function H is independent of $x(t)$ *and* $u(t)$.

## 2.0 Preliminaries

Let B denote the Banach space of continuous $R^n \times R^p$ - valued functions defined on $J = [0, t_1]$ with norm $\|(x,u)\| = \|x\| + \|u\|$ *where* $\|x\| = \sup\{|x(t)| : t \in J\}$ and $\|u\| = \sup\{|u(t)| : t \in J\}$. That is

$$B = C_n[0,t_1] \times C_p[0,t_1] \, where \, C_n$$

is the Banach space of continuous $R^n$ -valued function on $[0,t_1]$ with supremum norm. Consider the linear neutral volterra integroidifferential system with distributed delays of the form

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds + \int_{-h}^0 d_\theta H(t,\theta,x(t),u(t))u(t+\theta) \quad (2.1)$$

and the perturbed equation

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds + \int_{-h}^0 d_\theta H(t,\theta,x(t),u(t))u(t+\theta)$$
$$+ f(t,x(t),u(t)) \quad (2.2)$$

Here $x \in R^n$ and $u$ is $p-$dimensional control function with $u \in C_p[-h,t]$ and $H(t,\theta,x(t),u(t))$ is an $n \times p$ -dimensional matrix, continuous in $(t,x(t),u(t))$ for fixed $\theta$ and of bounded variation in $\theta$ on $[-h,0]$ for each $(t,x,u) \in J \times R^{n+p}$. The $n \times n$ matrix $A(t), C(t,s)$ and $G(t,s)$ continuous in there various arguments. The $n$ -vector function $f$ *and* $g$ respectively continuous and absolutely continuous. The integral is in the Lebesgue-Stieltjes sense which is denoted by the symbol $d\theta$.

***Definition* 2.1**

*The set* $Z(t) = \{x(t),u(t)\}$ *is said to be the complete state of system* (2.1) *at time t.*

***Definition* 2.2**

*The system* (2.1) *and* (2.2) *is said to be relatively controllable on* $J$ *if for every complete state* $Z(0)$ *and* $x_1 \in R^n$, *there exists a control function* $u(t)$ *defined on* $[0,t_1]$ *such that the solution* (2.1) *and* (2.2) *satisfies* $x(t_1) = x_1$

Using [13], the solution of (2.1) can be written as

$$x(t) = Z(t,0)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t,s)g(s)ds + \int_0^t Z(t,s)\left(\int_{-h}^0 d_\theta H(s,\theta,x(s))u(t+\theta)\right) ds \quad (2.3)$$

where $Z(t,s)$ and $\dot{Z}(t,s)$ are continuous matrices satisfying

$$\dot{Z}(t,s) - \int_0^t \dot{Z}(t,\tau)C(\tau,s)d\tau + C(t,s) = -Z(t,s)A(s) - \int_0^t Z(t,\tau)G(\tau,s)d\tau$$

and where $Z(t,t) = I$ and the solution of the nonlinear system (2.2) is given by

$$x(t) = Z(t,0)[x(0) - g(0)] + g(t) - \int_0^t \dot{Z}(t,s)g(s)ds$$
$$+ \int_0^t Z(t,s)\left[\int_{-h}^o d_\theta(s,\theta,x(s),u(s))u(t+\theta) + f(s,x(s),u(s))\right]ds \quad (2.4)$$

using the unsymmetric Fubini theorem as in [11], equation(2.3) can be written as

$$x(t) = Z(t,0)[x(0) - g(0)] + g(t) - \int_0^t \dot{Z}(t,s)g(s)ds + \int_{-h}^0 d_{H_\theta} \int_0^t Z(t,s-\theta)H(s-\theta,\theta,x,u)u_0(s)ds$$
$$+ \int_0^t\left[\int_{-h}^0 Z(t,s-\theta)d_\theta \overline{H}(s-\theta,\theta,x,u)\right]u(s)ds \quad (2.5)$$

where $d_{H_\theta}$ denote that the integration is in the Lebesgue-Stieltjes sense with respect to the variable $\theta \, in \, H$ and

$$\overline{H}(s,\theta,x,u) = \begin{cases} H(s,\theta,x,u) & for \ s \le t \\ 0 & for \ s > t \end{cases}$$

Let us define

$p(t) = Z(t,0)[x(0) - g(0)] + g(t) - \int_0^t \dot{Z}(t,s)g(s)ds$ and $q(t) = \int_{-h}^0 d_{H_\theta} \int_o^t Z(t,s-\theta)H(s-\theta,\theta,x,u)u_0(s)ds$

and the controllability matrix $W(0,t) = \int_0^t S(t,s)S^T(t,s)$ where the $T$ denotes the matrix transpose and

$$S(t,s) = \int_{-h}^0 Z(t,s-\theta)d_\theta \overline{H}(s-\theta,\theta,x,u)$$

then the equation (2.5) and (2.4) becomes

$$x(t)= p(t)+q(t)+\int_0^t S(t,s)u(s)ds \tag{2.6}$$

And $$x(t)= p(t)+q(t)+\int_0^t S(t,s)u(s)ds+\int_0^t Z(t,s)f(s,x(s).u(s)) \tag{2.7}$$

It is clear that $x_1$ can be obtained if there exists $x_1$ and u such that

$$u(t)=S^T(t_1,t)W^{-1}(0,t_1)\left[x_1 - p(t_1)-q(t_1)-\int_0^{t_1} Z(t_1,s)f(s,x(s),u(s))ds\right] \tag{2.8}$$

and $$x(t)= p(t)+q(t)+\int_0^t S(t,s)u(s)ds+\int_0^t Z(t,s)f(sa,x(s),u(s))ds \tag{2.9}$$

## 3.0 Main results

***Theorem* 3.1**

*System* (2.1) *is relatively controllable on the interval* J *iff W is nonsingular*

**Proof**

Assume $W$ is nonsingular. Let the control function u be defined on $J$ as

$$u(t)=S^T(t_1,t)W^{-1}(0,t_1)\left[x_1 - p(t_1)-q(t_1)-\int_0^{t_1} Z(t_1,s)f(s,x(s),u(s))ds\right]$$

then from (2.3), it follows that $x(t_1)=x_1$

Conversely, suppose that (2.1) is relatively controllable. In other to show that W is nonsingular, let us assume the contrary. Then, there exists a vector $v\neq 0$ such that $v^T W v=0$. It follows that

$$\int_0^{t_1} v^T S(t_1,s)v ds=0$$

therefore $v^T S(t,s)=0$ for $s\in J$. Consider the initial point, $x(0)=0$, and the final point $x_1 = v$. Taking $g=0$; since the system is relatively controllable there exists a control $u(t)$ on $J$ that steers the responds to $x_1 = v$ at $t=t_1$, that is $x(t_1)=v=\int_0^{t_1} S(t_1,s)u(s)ds$ and hence $v^t v=\int_0^{t_1} v^T S(t_1,s)u(s)ds=0$, this is a contradiction for $v\neq 0$. Hence $W$ is nonsingular.

***Theorem* 3.2**

*Let the continuity conditions on the matrices G,S,f and the continuously differentiability of C,g be satisfied for the systems with the following additional condition that measurable function* $\phi_j : R^{n+p}\to R^+$ *and* $L^1$ *function* $\alpha_j : J\to R^+$, $j=1,2.,\cdots,q$ *be such that* $\left|f(t,x,u)\right|\le\sum_{j=1}^{q}\alpha_j(t)\phi_j(x,u)$ *for every* $(t,x,u)\in J\times R^{n+p}$*, then the relative controllability of* (2.1) *implies the relative controllability of* (2.2) *if*

$$\lim_{r\to\infty}\sup\left(r-\sum_{j=1}^{q} c_j \sup\left[\phi_j(x,u):\left\|(x.u)\right\|\le r\right]\right)=\infty \tag{3.1}$$

**Proof**

Let B be a Banach space and define the nonlinear continuous operator $T: B\to B$ by $T(x,u)=(y,v)$ as follows

$$v(t)=S^T(t_1,t)W^{-1}(0,t_1)\left[x_1 - p(t)-q(t_1)-\int_0^t Z(t,s)f(s,x(s)u(s))ds\right] \tag{3.2}$$

and $$y(t)= p(t)+q(t)+\int_0^t S(t,s)v(s)ds+\int_o^t Z(t,s)f(s,x(s),u(s))ds \tag{3.3}$$

Let us introduce the following notations

$a_1=\sup\left\|S^T(t_1,t)\right\|\ t\in J$, $a_2=\left\|W^{-1}(o,t)\right\|$, $a_3=\sup\left[\left|p(t_1)\right|+\left|q(t_1)\right|+\left|x_1\right|;\ t_1\in J\right]$, $a_4=\sup\left[\left\|S(t,s)\right\|;\ t\in J\right]$

$\left\|\alpha_j\right\|=\int_0^{t_1}\left|\alpha_j\right|ds$, $C_{1j}=6ba_1a_2a_4\left\|\alpha_j\right\|$, $C_{2j}=6a_4\left\|\alpha_j\right\|$, $C_j=\max\left|C_{1j},C_{2j}\right|$, $d_1=6a_1a_2a_3b$, $d_2=6a_3$, $b=\max\{(t_1)a_4,1\}$, $d=\max\{d_1,d_2\}$.

Let $\psi_j(r)=\sup\{\phi_j(x,u):\left\|(x,u)\right\|\le r\}$. Since (3.1) holds there exists $r_0>0$ such that $\sum_{j=1}^{q}C_j\psi_j(r_0)+d\le r_0$

Now let $B_{r_0}=\{(x,u)\in B:\left\|(x,u)\right\|\le r_0\}$, if $(x,u)\in B_{r_0}$ then from (2.9) and (3.1) we have

$$\left|v(t)\right| \le a_1 a_2\left[a_3 + a_4\int_0^t \left\|Z(t,,s)\right\|\sum_{j=1}^{q}\alpha_j(s)\phi_j(x(s),u(s))ds\right] \le \frac{d_1}{6b} + \sum_{j=1}^{q}\frac{C_{1j}}{b}\psi_j(r_0) \le \frac{1}{6b}\left[d + \sum_{j=1}^{q}C_j\psi_j(r_0)\right] \le \frac{1}{6b}r_0 \le \frac{r_0}{6}$$

also $$\left|y(t)\right| \le a_3 + a_4(t_1)\left\|v\right\| + a_4\int_0^{t_1}\left\|Z(t,s)\right\|\sum_{j=1}^{q}\alpha_j(s)\phi_j(x(s),u(s))ds \le a_3 + \sum_{j=1}^{q}a_4\left\|\alpha_j(s)\right\|\psi_j(r_0) + b\left\|v\right\|$$

$$\le b\left\|v\right\| + \frac{d}{6}\sum_{j=1}^{q}\frac{C_j}{6}\psi_j(r_0)\ b\left\|v\right\| + \frac{r_0}{6} \le \frac{r_0}{6} + \frac{r_0}{6} = \frac{r_0}{3}$$

since $\left|v(t)\right| \le \frac{r_0}{9}$, $\left|y(t)\right| \le \frac{r_0}{3}$ we have proved that $B_{r_0} \supset T(B_{r_0})$. Hence $T\ maps\ B_{r_0}$ into itself by our assumptions, the operator $T$ is continuous. Further it is easy to see that $T(B_{r_0})$ is equicontinuious for all $r > 0$ Kantrovich [9]. By the Ascoli-Arzela theorem $T(B_{r_0})$ is relatively compact. Since $B_{r_0}$ is closed, bounded and convex, the schauder fixed point theorem guarantees that $T$ has a fixed point $(x,u) \in B_r$, such that $T(x,u) = (x,u)$. It follows that for $(x,u) = (y,u)$, we have

$$x(t) = p(t) + q(t) + \int_0^t S(t,s)u(s)ds + \int_0^t Z(t,s)f(s,x(s),u(s))ds \qquad (3.4)$$

Thus the solution (2.8) and (2.9) exists. Hence the system is relatively controllable on *J*.

## 4.0 Conclusion:

Using Schauder's fixed point theorem relative controllability on $[0,t_1]$ of a class of nonlinear neutral integrodifferential system with distributed delays in the control has been studied. An explicit expression is given for transferring a given state to a desired state on the interval $[0,t_1]$ using appropriate control, while in Balachandran and Dauer [3] appropriate control and its corresponding solutions are define by an integral equation and solved.

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