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On the convergence profile of a discretized scheme for a two-dimensional constrained optimal control problem

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Abstract

The convergence profile of a discretized schame for an optimal control problem constrained by ordinary differential equation with matrix coefficients is examined. Various penalty parameters are considered for the penalty function method. It is observed that convergence is exhibited for these penalty parameters after certain number of iterations with a predetermined interval of convergence.

1.0 Introduction.

Previous works, [Olorunsola, 1991] and [Ibiejugba, 1982], have compared the results of various numerical methods of optimal control problems and the analytic results. Here, consideration is given only to the convergence profile of the discretized scheme for optimal control problems constrained by differential equation with matrix coefficients, since earlier paper, by [Olorunsola and Olotu, Ref, 8], has investigated similar problem with real coefficients whose theoretical approach is employed in this work. Also, an associated matrix operator for the bilinear form expression is constructed using a modified form of Ibiejugba's control operator based on Fletcher and Reeve's idea on function minimization [1952]. Sampled problem was examined to determine the convergence profile for various penalty paramater constants on the penalty function method. Within a region of uncertainty predetermined by a given constant.

Now a generalized two-dimensional constrained optimal control problem is considered thus;

2.0 Generalized problem

$$Min\int_0^Z \left(x(t)^T Px(t)+u(t)^T Qu(t)\right)dt$$

Subject to $\dot{x}(t)=Ax(t)+Cu(t) \quad x(0)=x_0 \quad 0\le t\le Z$ (2.1)

where, $x(t)$, $u(t)\varepsilon R^2$, $x(t)^T, u(t)^T$ denote the transpose of $x(t)$ and $u(t)$ and respectively P,Q are 2 by 2 symmetric matrices. A and C are 2 b y 2 matrices not necessarily symmetric.

The constrained problem (2.1) can be turned into unconstrained problem via the penalty method [Ref, 2].

$$\langle Z, AZ\rangle_{\overline{K}} = Min_{(x,u,\mu)}\int_0^Z \left\{x(t)^T Px(t)+u(t)^T Qu(t)+\mu\left\|\dot{x}(t)-Ax(t)-Cux(t)\right\|^2\right\} \qquad (2.2)$$

$\mu>0$, the penalty constant.

3.0 Discretization

By discretizing (2.2), we have

$$\dot{X}(t) = Ax(t) + Cu(t)$$
$$\left(x(k+1) - x(k)\right)/\Delta_k = Ax_k(t_k) + Cu_k(t_k) \qquad (3.1)$$
$$X(0) = 0$$

We then have the discretized generalized problem in the form;

$$\min J = \sum_{k=0}^{n} \Delta_k \left(x(t)^T Px_k\ (t_k) + u(t)^T Qu_k\ (t_k)\right)$$
$$\text{subject to } \left(x(k+1) - x(k)\right)/\Delta_k = Ax_k(t_k) + Cu_k(t_k)$$
$$x(0) = 0 \qquad (3.2)$$

4.0 Penalty method's application

$$MinJ(x,u,\mu) = \sum_{k=0}^{n} \left\{\Delta_k \left(x(t)^Y Px_k\ (t_k) + u(t)^T Qu_k\ (t_k)\right) + \mu[x_{K+1}(t_{k+1}) - x_k(t_k) - \Delta_k Ax_k(t_k) - C\Delta_k u_k(t_k)]^2\right\}$$

$$= \sum_{k=0}^{nn} \left\{ \begin{array}{l} x_k^T(t_k)[P\Delta_k + \mu + \mu A^T \Delta_k^T A + 2\mu A\Delta_k]x(t_k) + u_k(t_k)^T[Q\Delta_k + \mu C^T \Delta_k^T \Delta_k c\ ]u(t_k) + \mu x_{k+1}^T(t_k)x(t_{k+1}) \\ + x_k^T(t_k)[2C^T \Delta_k^T \mu + 2C^T \Delta_k^T \Delta_k A\mu]u_k(t_k) \\ + x_{k+1}^T(t_k)[-2\mu - 2\mu A\Delta_k]x_k(t_k) + x_{k+1}^T(t_k)[-2\mu C\Delta_{k]}]u_k(t_k) \end{array} \right\} \qquad (4.1)$$

Let $Z_k = \begin{pmatrix} x_k(t_k) \\ u_k(t_k) \end{pmatrix}$, and $y_k(t_k) = x_{k+1}(t_k)$

$$Mk = P\Delta_k + \mu + \mu A^t \Delta_k^t A + 2\mu A\Delta_k$$
$$Nk = Q\Delta_k + \mu C^T \Delta_k^T \Delta_k c$$
$$Bk = 2C^T \Delta_k^T \mu + 2C^T \Delta_k^T \Delta_k A\mu$$
$$Pk = -2\mu - 2\mu A\Delta_k$$
$$Ak = -2\mu C\Delta_k$$

Now, (4.1) becomes

$$\sum_{k=0}^{n} \left\{Mkx_k^2(t_k) + N_k u_k^2(t_k) + \nu y_k^2(t_k) + Akx_k(t_k)u_k(t_k) + Bky_k(t_k)x_k(t_k) + Pky_k(t_k)u_k(t_k)\right\} \qquad (4.2)$$

5.0 Construction of operator G

Writing in bilinear form we have,

$$\langle Z_{k1}(t_k), GZ_{k2}(t_k)\rangle = \sum_{k=0}^{n} \{M_k x_{k1}^T(t_k)x_{k2}(t_k) + Nk_k u_{k1}^T(t_k)u_{k2}(t_k) + y_{k1}^T(t_k)y_{k2}(t_k)\mu$$
$$+ Akx_{k1}^T(t_k)u_{k2}(t_k) + Aku_{k1}^T(t_k)x_{k2}(t_k) + Bky_{k1}^T(t_k)x_{k2}(t_k) + Bky_{k2}^T(t_k)x_{k1}(t_k) \qquad (5.1)$$
$$+ Pky_{k1}^T(t_k)u_{k2}(t_k) + Pky_{k2}^T(t_k)u_{k1}(t_k)\}$$

where $GZ_{K2}(t_k) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} x_{k2} \\ u_{k2} \end{pmatrix} = \begin{pmatrix} G_{11}x_{k2} + G_{12}u_{k2} \\ G_{21}x_{k2} + G_{22}u_{K2} \end{pmatrix}$

Involving the derivative of *x* and simplifying, we have

$$\langle Z_{K1}(t_k), GZ_{K2}(t_k)\rangle_H = \sum_{k=0}^{n}\{M_k x_{K1}(t_k)x_{K2}(t_k) + N_k u_{K1}(t_k)u_{K2}(t_k) + \mu\Delta_K{}^T \dot{x}_{K1}(t_k)\dot{x}_{K2}(t_k)\Delta_k + \mu\Delta_K{}^T \dot{x}_{K1}(t_k)x_{K2}(t_k)$$

$$+\mu\Delta_K\ x_{K1}{}^T(t_k)\dot{x}_{K2}(t_k) + \mu x_{K1}{}^T(t_k)x_{K2}(t_k) + \lambda B_k \dot{x}_{K1}{}^T(t_k)x_{K2}(t_k) + B_K x_{K1}{}^T(t_k)x_{K2}(t_k) + B_K\Delta_K{}^T x\dot{x}_{k21}(t_k)x_{K1}(t_k) \quad (5.2)$$

$$+B_K x_{K2}{}^T(t_k)x_{K1}(t_k) + P_K\Delta_K{}^T \dot{x}_{K1}(t_k)u_{K2}(t_k) + P_K x_{K1}{}^T(t_k)u_{K2}(t_k) + P_K\Delta_K{}^T \dot{x}_{K2}(t_k)u_{K1}(t_k) + P_K x_{K2}{}^T(t_k)u_{K1}(t_k)$$

$$+u_{k1}{}^T(t_k)x_{k2}(t_k)\alpha_k + u_{k2}{}^T(t_k)x_{k1}(t_k)\alpha_k\}$$

Setting $u_{K2}(t_k) = 0$, in (5.2) we have

$$\begin{pmatrix} G_{11}x_{K2} \\ G_{21}x_{K2} \end{pmatrix} = \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}$$

$$\sum_{k=0}^{n}\{\ x_{K1}{}^T(t_k)[x_{K2}(t_k)M_k + \mu x_{K2}(t_k) + 2B_K x_{K2}(t_k) + \mu\Delta_K\ \dot{x}_{K2}(t_k) + B_K\Delta_K{}^T \dot{x}_{k2}(t_k)] +$$

$$+\dot{x}_{K1}{}^T(t_k)[x_{K2}(t_k)(\Delta_k{}^T B_k + \Delta_k{}^T\mu) + \dot{x}_{k2}(t_k)\mu_k{}^T\Delta_k] + u_{k1}{}^T(t_k)[x_{k2}(t_k)(P_k + A_k) + \Delta_k{}^T \dot{x}_{kk2}(t_k)P_k] \quad (5.3)$$

$$= \sum_{k=0}^{n}\{x_{K1}{}^T(t_k)V_{11}(t_k) + \dot{x}_{K1}(t_k)\dot{V}_{11}(t_k) + u_{K1}(t_k)V_{21}\} \text{ Define} \quad (5.4)$$

$$\Omega(t_k) = (M_K + \mu + 2B_K)x_{K2}(t_k) + (\mu\Delta_K + B_K\Delta_K{}^T)\dot{x}_{K2}(t_k) \quad (5.5)$$

$$f(t_k) = \mu\Delta_K{}^T\Delta_k\dot{x}_{K2}(t_k) + (\mu\Delta_K{}^T + B_K\Delta_k{}^T)x_{K2}(t_k) \quad (5.6)$$

So, $\quad A_{21}u_{k1}(t_k) = x_{k2}(t_k)(P_k + A_k) + \Delta_k{}^T \dot{x}_{kk2}(t_k)P_k]$ (5.7)

To obtain the component *A*11 x *K*1 (*tk*) =V11 (*tk*), $\Omega(t_k) - V_{11}(t_k)$ and $f(t_k) - \dot{V}_{11}(t_k)$ are both continuous functions on [*0,T*] , i.e. $\Omega(t_k)$ is a function of $x_{k2}(t_k)$ and $\dot{x}_{k2}(t_k)$ which are both continuous. So also, $f(t_k)$ is a function of $\dot{x}_{k2}(t_k)$ and $x_{k2}(t_k)$.. Hence the difference of two continuous functions is continuous. And choosing $x_{K1}(\bullet) \in D[0,T] \ni x_{K1}(0) = x_{K1}(T) = 0.$

We then have , by Gelfand and Formin,

$$\int_0^T \{x_{K1}(t_k)[\Omega(t_k) - V_{11}(t_k)] + \dot{x}_{K1}(t_k)[f(t_k) - \dot{V}_{11}(t_k)]\}dt_k = 0 \quad (5.8)$$

Now, $f(t_k) - \dot{V}_{11}(t_k)$ is continuously differentiable on $[0,T]$ with

$$\frac{d}{dt}[f(t_k) - \dot{V}_{11}(t_k)] = \Omega(t_k) - V_{11}(t_k) \quad (5.9)$$

$$\dot{f}(t_k) - \ddot{V}_{11}(t_k) = \Omega(t_k) - V_{11}(t_k) \text{ or } \ddot{V}_{11}(t_k) - V_{11}(t_k) = \dot{f}(t_k) - \Omega(t_k).$$

This is a second order differential equation ;

$$\ddot{V}_{11}(t_k) - V_{11}(t_k) = q(t_k) = \dot{f}(t_k) - \Omega(t_k) \quad (5.10)$$

with the initial conditions $V_{11}(0) = p_0$ and $\ddot{V}_{11}(0) = r_0$.

Solving (5.10) by Laplace transform and letting

$$L\{V_{11}(t_k)\} = \hat{V}_{11}(s), \quad L\{q(t_k)\} = Q(s)$$

We have $\quad S^2\hat{V}(s) - p_0 s - r_0 - \hat{V}_{11}(s) = Q(s), \ \hat{V}_{11}(s) = \frac{Q(s)}{s^2 - 1} + \frac{p_0 s}{s^2} + \frac{r_0}{s^2 - 1}$

and taking the inverse of Laplace transform, we have

$$V_{11}(t_k) = \int_0^T q(s_k)\sinh(t_k - s_k)ds_k + p_0\cosh(t_k) + r_0\sinh(t_k) \quad (5.11)$$

But, $\quad \Omega(T) - V_{11}(T) = 0$ (5.12)

$$\Omega(0) - V_{11}(0) = 0 \quad (5.13)$$

$\Omega(0) = \rho_0\ \ \Omega(0) = (M_k + \mu + 2B_k)x_{k2}(0) + (\mu\Delta_k + \Delta_k{}^T B_k)\dot{x}_{k2}(0) = \rho_0$

From (5.12) $\Omega(T) = V_{11}(T)$

$$V_{11}(T) = \int_0^T q(s_k)\sinh(T - s_k)ds_k + [(M_k + \mu + 2B_k)x_{K2}(0) + (\mu\Delta_K + \Delta_K{}^T B_k)\dot{x}_{K2}(0)]\cosh(T) + \tau_0 \sinh(T)$$

$$= [(M_k + \mu + 2B_k)\dot{x}_{K2}(T) + (\mu\Delta_K + \Delta_K{}^T B_k)\dot{x}_{K2}(T)]$$

Therefore,

$$\tau_0 = \frac{1}{\sinh(T)}\langle -\int_0^T q(s_k)\sinh(T - s_k)ds_k - [(M_k + \mu + 2B_k)x_{K2}(0) + (\mu\Delta_K + \Delta_K{}^T B_k)\dot{x}_{K2}(0)]\cosh(T) + \tag{5.14}$$

$$[(M_K + \mu + 2B_K)\dot{x}_{K2}(T) + (\mu\Delta_K + \Delta_K{}^T B_k)\dot{x}_{K2}(T)]\rangle$$

But $q(t_K) = \dot{f}(t_k) - \Omega_K(t_k)$

$$\int_0^T f(\dot{s}_K)\sinh(t_K - s_k)ds_k = -\sinh(T)f(0) + \int_0^T f(\dot{s}_K)\cosh(t_K - s_k)ds_k \tag{5.15}$$

Now,

$$\int_0^T q(s_k)\sinh(T - s_k)ds_k = -\sinh T[x_{k2}(0)(\Delta_k{}^T \mu\Delta_k{}^T B_k) + \dot{x}_{k2}(0)(\mu\Delta_k{}^T\Delta_k)]$$

$$+ \int_0^T [x_{k2}(s_{jk})(\Delta_k{}^T \mu\Delta_k{}^T B_k) + \dot{x}_{k2}(s_k)(\mu\Delta_k{}^T\Delta_k)]\cosh(T - s_K)ds_K \tag{5.16}$$

$$- \int_0^T \{(M_K + \mu + 2B_K)x_{K2}(s_k) + (\Delta_k\mu + \Delta_K{}^T B_k)\dot{x}_{K2}(s_k)\}\sinh(T - s_k)ds_k$$

From (5.14) and (5.15), we have

$$\tau_0 = \frac{1}{\sinh T}\{[(M\alpha_K + \mu + 2B)x_{K2}(T) + (\Delta_k{}^T\mu + \Delta_k B_k)\dot{x}_{K2}(T)] - [(M_K + \mu + 2B_K)x_{K2}(0)$$

$$+ (\Delta_k\mu + \Delta^T B_K)\dot{x}_{K2}(0)cshT\} - \frac{1}{\sinh T}\{-\sinh T[(\Delta_K{}^T\mu + \Delta_K{}^T B_k)x_{K2}(0) + \dot{x}_{K2}(0)(\mu\Delta_k{}^T\Delta_k)]$$

$$+ \int_0^T \{[(\Delta_K{}^T\mu + \Delta_K{}^T B_k)x_{K2}(s_k) + \dot{x}_{K2}(s_k)(\mu\Delta_k{}^T\Delta_k)]\}\cosh(T - s_k)ds_k - \int_0^T \{(M_K + \mu + 2B_k)x_{K2}(s_k)$$

$$+ (\Delta_k{}^T\mu + \Delta_k{}^T B_k)\dot{x}_{K2}(s_k)\}\sinh(T - s_k)ds_k\} \tag{5.17}$$

$$V_{11}(t_k) = G_{11}(t_k) = \tau_0 \sinh(t_k) + [(M_k + \mu + 2B_k)x_{K2}(0) + (\Delta_k\mu + \Delta_k{}^T B_k)\Delta_K \dot{x}_{K2}(0)]\cosh t_k$$

$$-\sinh T\{x_{k2}(0)(\Delta_k{}^T\mu + \Delta_k{}^T B_k) + \dot{x}_{K2}(0)(\mu\Delta_k{}^T\Delta_k\} + \int_0^T \{x_{k2}(s_k)(\Delta_k{}^T\mu + \Delta_k{}^T B_k) + \dot{x}_{K2}(s_k)(\mu\Delta_k{}^T\Delta_k\}csh(T - s_k)ds_k$$

$$- \int_0^T \{(M_K + \mu + 2B_K)x_{K2}(s_k) + (\Delta_k{}^T\mu + \Delta_k B_K)\dot{x}_{K2}(s_k)\}\sinh(T - s_k)ds_k \tag{5.18}$$

In equation (2.1), setting

$x_{K2}(t_k) = 0 \to \dot{x}_{K2}(t_k) = 0$

We have $\begin{pmatrix} G_{12}u_{k2}(t_k) \\ G_{22}u_{k2}(t_k) \end{pmatrix} = \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix}$

$$\langle Z_{K1}, AZ_{K2}(t_k)\rangle_H = \sum_{k=0}^n \{ u_{K1}{}^T(t_k)u_{K2}(t_k)N_k + \Delta_k{}^T \dot{x}_{K1}(t_k)u_{K2}(t_k)P_k + x_{K1}{}^T(t_k)u_{K2}(t_k)P_k$$

$$+ u_{K2}{}^T(t_k)x_{k1}(t_k)A_k\}$$

$$= \sum_{k=0}^n \{x_{K1}{}^T(t_k)[u_{K2}(t_k)P_k \quad + u_{K2}(t_k)A_k] + \dot{x}_{K1}{}^T[u_{K2}(t_k)\Delta_k P_k] + u_{K1}{}^T(t_k)u_{K2}(t_k)N_k\}$$

$$= \sum_{k=0}^n \{x_{K1}{}^T(t_k)V_{12}(t_k) + \dot{x}_{K1}{}^T V_{12}(\dot{t}_k) + u_{K1}{}^T(t_k)V_{22}(t_k)\} \tag{5.19}$$

So, $V_{22}(t_k) = G_{22}u_{K2}(t_k) = u_{K2}(t_k)N_k$

Again define

$g(t_k) = u_{K2}(t_k)(P_k + A_k)$ and $h(t_k) = u_{K2}(t_k)\Delta_k{}^T Pg(t_k) - V_{12}(t_k)$ *and* $h(t_k) - \dot{V}_{12}(t_k)$ are continuous functions on $[O,T]$.

As before, $$V_{12}(t_k) = \int^T q(t_k)\sinh(t_k - s_k)ds_k + e_0 csht_k + l_0 \sinh t_k \tag{5.20}$$

where $e_0 = g(0) = u_{K2}(0)(P_k + A_k)$ *and* $V_{12}(T) = g(T)$

So, $l_0 = [g(T) - \int_0^T q(s_k)\sinh(T - s_k)ds_k - g(0)\cosh T]/\sinh T = [u_{K2}(T)(P_k + A_k) - \int_0^T q(s_k)\sinh(T - s_k)ds_k$
$- u_{K2}(0)(P_k + A_k)cshT]/\sinh T$

But $q(s_k) = \dot{h}(s_k) - g(s_k)$. Therefore,

$$\begin{aligned}\int_0^T q(s_k)\sinh(T - s_k)ds_k &= -\sinh(T)U_{k2}(0)\Delta_k{}^T P_k + \int_0^T U_{k2}(s_k)\Delta_k{}^T P_k Cosh(T - s_k)ds_k \\ &- \int_0^T U_{k2}(s_k)(P_k + A_k)\sinh(T - s_k)ds_k\end{aligned} \tag{5.21}$$

From (5.20) and (5.21),

$$\begin{aligned}V_{12}(t_k) = G_{12}U_{k2}(t_k) &= e_o \sinh(t_k) + u_{k2}(0)(P_k + A_k)\cosh(t_k) - \sinh(T)(u_{k2}(0)\Delta_k{}^T P_k) \\ &+ \int_0^T (u_{k2}(s_k)\Delta_k{}^T P_k)\cosh(T - s_k)ds_k - \int_0^T (u_{k2}(s_k)(P_k + A_k)\sinh(T - s_k)ds_k\end{aligned} \tag{5.26}$$

6.0 Data and analysis of sampled problems

***Example* 5.1**

Minimize $\int_0^1 \{x^T(t)Px(t) + u^T Qu(t)\}$

Such that $\dot{x}(t) = Ax(t) + Cu(t), \quad x(0) = x_0$

where,

$$P = \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, A == \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, x_0 = \begin{pmatrix} 1 \\ .5 \end{pmatrix}, u_0 == \begin{pmatrix} 1 \\ .5 \end{pmatrix},$$

conform to the specified format in terms of symmetry or not,

Table 1: Shows the convergence profile of Example 1

Penalty Constant	Iteration	Objective Function	Constraint Satisfaction	Penalty Function
1	2	3	4	5
μ=.0001	1	5	10	5.001
	2	3.6554451	7.115152	3.655163
	3	3.568452	7.05019	3.569159
μ=.0002	1	5	10	5.001999
	2	3.248526	5.669127	3.244966
	3	2.225528	3.799845	2.2262288
	4	1.931005	3.392155	1.931683
μ=.0003	1	5	10	5.003
	2	3.144416	5.39738	3.145815
	3	2.079541	3.393059	2.080559
	4	1.806033	2.9870036	2.08693
μ=.0004	1	5	10	5.004001

	2	3.211735	5.59364	3.213973
	3	2.161946	3.662971	2.1634411
	4	1.8850631	3.207022	1.851913
μ=.0005	1	5	10	5.005001
	2	3.244694	5.681146	3.247535
	3	2.20644	3.787466	2.206568
	4	1.89938	3.31896	1.881597

7.0 Conclusion and Comments

In Table 1, above, for μ=.0001, the iteration exhibits some convergence and terminates at the third iteration as seen in columns 3,4 and 5 for the objective function, constraint satisfaction and penalty function respectively.

For other various parameters $.0002 \leq \mu \leq .0005$, the iterates exhibit better convergence profile and terminate at the fourth iteration as seen particularly in column 3 where the objective function value starts at 5 for each parameter and terminates at 1.931005, 1.806033, 1.88631 and 1.89938 respectively for μ=.0002, μ=.0003, μ=.0004, μ=.0005. It is observed that the lowest objective function value 1.806053 occurs at the fourth iteration for μ=.0003. This indicates that the best approximate optimizer is attained at this parameter.

Based on the above finding , it is implied that the developed scheme exhibits a convincing convergence profile for each considered parameter as exhibited in the objective function values, constraint values and penalty function values within a region of uncertainty predetermined by the given bound -.8258 and +.8258, picked arbitrarily.

. In fact, this is a contribution to knowledge in the area of optimal control problems constrained by differential equation with matrix coefficients.

References

[1] Reeves, C. M. ; Function mimimization by conjugate gradiends, Computer Journal, Electronic Computing Laboratory, The university , Leeds 2, Vol.7, pg. 149-154, 1964.

[2] Glad, S.T.;A combination of the penalty function and the multiplier methods for solving optimal control problems,Journal of optimization theory and application. Vol. 28, N3, pg. 330-33, 1979.

[3] Hesteness, M.R.; Multiplier and gradient methods , Journal of optimization theory and applications Vol. 4, pg. 33-42, 1969

[4] Hesteness, M.R. and Stiefel, E. ; Method of conjugate gradients for solving linear systems, Journal of research and National Burrough standard , Vol. 4, pg. 409-436, 1952

[5] Ibiejugba, M.A.,The construction of control operator , An. Stiint. University. ‘A1.I. Cuza’., Iasi XXVIII ,pg, 45-48, 1982.

[6] Olorunsola, S.A.; The effect of large penalty constants on the convergence rate of the penalty method , Journal of optimization , Vol. Pg.35-69

[7] Shannon, D.F. On the convergence of new conjugate gradient algorithm , SIAM, Journal of Numerical Analysis . Vol 15, N6, pg 1247-1257 , 1978

[8] Olorunsola, S.A and Olotu, O A discretized algorithm for the solution a constrained , continuous quadratic control problem, Journal of the Nigeria Association of Mathematical Physics, University of Benin, Benin, vol. 8. Pg.295-300 2004