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Criteria for exponential asymptotic stability in the large of perturbations of linear systems with unbounded delays.

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Abstract

The purpose of this study is to provide necessary and sufficient conditions for exponential asymptotic stability in the large and uniform asymptotic stability of perturbations of linear systems with unbounded delays. A strong relationship is established between the two types of asymptotic stability. It is found that if the exponential estimate of the solution of a system tends to zero as $t \to \infty$ the system is said to be uniformly asymptotically stable. But if the solution of a system approaches the origin faster than any exponential function, then the system is said to be exponentially asymptotically stable. Utilizing the exponential estimate of the solution, stability criteria for the linear part of our system of interest is derived. With enough smoothness conditions on the perturbation function, and appeal made to Lyapunov's stability results and some Gronwall-type inequalities the required stability results are established for the linear perturbation.

Keywords: Exponentially asymptotic stability, linear perturbation, stability in the large, exponential estimate.

1.0 Introduction.

Stability theory begins with the problem of showing that solutions, which start near the trivial solution, stay near it for all future times. Efforts at tackling this problem are vast in the literature. Various stability results for the time varying ordinary homogeneous system given by

$$x(t) = A(t)x(t) \tag{1.1}$$

where $A(t)$ is n x n matrix function, are conveyed in [8] [9]. In [3], [8] computable criteria for the stability of the autonomous system.

$$x(t) = Ax \tag{1.2}$$

are presented including solved examples and applications of stability in science and technology. One of these results is the criterion for asymptotic stability of system (1.2), which is the requirement that all the roots of the characteristic equation of (1.2) have negative real parts. Kartsatos in [12] has spelt out a set of conditions for stability, uniform stability, uniform asymptotic stability of system (1.1). These conditions are based on the integral boundedness of the fundamental matrix solution of the given system(1.1) and the measure of measurable sets. However, the computation of the fundamental matrix solution of most systems is laborious and difficult to obtain. The characteristic equations and fundamental matrix solutions of functional differential equations are in most cases exponential formulations that are difficult to simplify and hence serve very little practical purpose (see [17]). To overcome this difficulty, the use of Lyapunov functions are introduced in proving conditions for the stability of ordinary and functional differential systems: The use of these positive definite functions followed the pioneering work of Alexsandr Mikhailovich Lyapunov in a monograph published in Russia in 1892. The methods of Lyapunov revolve around the notion that in a stable system the total energy in the system would be a minimum at the equilibrium point [17]. It is this total energy, that is described as the Lyapunov function. Burton in [3] stated that, if the derivative of the Lyapunov function is negative the non-trivial solution would have a precise bound. This is the condition that guarantees the asymptotic stability of the ordinary systems, (see [14]). Besides, the ordinary differential equations we have equations that incorporate delays. These equations are called functional differential equations. Hale [10] identifies two main types: - Delay differential equations and Neutral differential equations.

Delay differential equations are those equations whose derivative of the state is expressed in terms of the present as well as the past states. They are of form:

$$x(t) = f(t, x_t),\ x_{to} = \phi$$

Neutral systems are equations whereby the derivative of the functional difference operator $D(t, x_t)$ is expressed in terms of the past and present states. They are of the form

$$\frac{d}{dt}D((t, x_t)) = f(t, x_t);\ x_t = \phi$$

Functional differential equations are natural models for most real life problems. They have been found useful in the study of global economic growth in [5] with resultant emergence of universal economic laws. In the same work, it is reported that neutral systems are of immense importance to systems planners in the control of fluctuations of currents in loss less transmission lines. There are applications of functional differential systems in Biology, Physics Engineering (see Burton [3]).

The growing importance of functional differential equations has attracted the attention of researchers who have provided conditions for the existence and uniqueness of solutions of the equations. These results are summarized by Caratheodory's conditions for the existence and uniqueness of solutions of functional differential equations.

Research has recently extended to the investigation of the stability of functional differential equations (see [5] and [10]). Computable criteria for the stability of delay systems exist in the literature, see [7].

Lyapunov's methods have been extended to investigations on stability of functional differential systems by scholars among whom are Cheban [4], Chukuw [5] Hale [10], Cruz and Hale [7]. In [13], Kolomanovskii and Myshkis presented Lyapunov's direct and energy methods with appreciable instructional dexterity and force. Hale [10] and Cruz and Hale [7] provided illuminating examples that concretize and illustrate these methods. Banks in [1] obtained variation of constant formulae for the solutions of functional differential equations Kolomanovskii and Myshkis [13] and quite recently, Zhang [18] developed a new formulae for constructing Lyapunov functions and functionals for delay equations. Perturbation theorems for the stability of linear differential equations are presented in [11, 16].

Chukwu [5] has improved on the results in [7] and [10] to provide conditions for uniform stability, asymptotic stability and exponential asymptotic stability in the large; making distinction between the two. The present effort is to provide analogous results of Chukwu for perturbations of delay systems with infinite and unbounded delays.

2.0 Notation and Preliminaries

Consider the linear system with unbounded delays

$$x(t) = L(t, x_t) \qquad (2.1)$$

and its perturbation

$$x(t) = L(t, x_t) + f(t, x_t) \qquad (2.2)$$

$$L(t, x_t) = \sum_{k=0}^{\infty} A_k x(t - w_k) + \int_{-\infty}^{0} A(t, \vartheta) x(t+\theta) d\vartheta$$

We reduce systems (2.1, 2.2) to the systems with bounded delays by the following analysis. We start by rewriting part of (2.1) in the form.

$$L(t, x_t) = \lim_{p\to\infty}\sum_{k=0}^{p} A_k x(t - w_k) + \lim_{M\to\infty}\int_{-M}^{0} A(t+\vartheta) x(t+\theta) d\vartheta,\ M \succ 0$$

We assume that the limits exist giving the following partial sums: $\lim_{p\to\infty}\sum_{k=0}^{p} A_k x(t - w_k)$ exists and is finite Hence the sum of the infinite terms exists and is taken as $\sum_{k=0}^{p} A_k x(t - w_k),\ p \prec \infty$.

In like manner, we assume $\lim_{M\to\infty}\int_{-M}^{0} A(t+\vartheta)x(t+\theta)\,d\vartheta$ exists and is also finite, thereby establishing the finiteness of the indefinite integral

$$\int_{-\infty}^{0} A(t,\vartheta)x(t+\theta)\,d\vartheta$$

Hence system (2.2) becomes

$$\dot{x}(t) = L(t,x_t) + f(t,x_t)) \qquad (2.2)$$

$$x_t = \phi,\ \phi \in C([-h,0],E^n)$$

where

$$L(t,x_t) = \sum_{k=1}^{p} A_k x(t-w_k) + \int_{-k}^{0} A(t,\vartheta)x(t+\theta),\ h \succ 0$$

$$L(t,x_t) = \int_{-h}^{0} d\vartheta\eta(t,\vartheta)x(t+\theta) \qquad (2.3)$$

is satisfied almost everywhere. The integral is in the Lebesgue Stieltijes sense with respect to θ. $L(t, \phi)$, is continuous in t, linear in ϕ. A_k, $A(t, \theta)$ are n x n matrices. $\eta(t, \theta)$ is a matrix of bounded variation, with

$$\text{var}_{[-h,0]}\ \eta(t,\theta) \le M(t) \text{ where } M(t) \in L_1(t_0, t_1], E)$$

L_1 is the space of integrable functions. Let E be the real line $(-\infty,\infty)$ and $J = [t_o, t_1]$; $t_1 > t_o$ is a subset of E. For an integer n, E^n is the Euclidean space of n tuples with the Euclidean norm $|.|$. The state space $C = C([-h, 0], E^n)$ is the Banach space of continuous functions, the delay $h > 0$. For the function $x:[-h, t] \to E^n$,

$$x_t(s) = x(t+s) \text{ for } t > 0 \text{ and } s \in [-h,0]$$

The function $f\ J \times C \to E^n$ is continuous.

3.0 Definition of Terms

***Definition* 3.1**

The trivial solution, $x = 0$ *of* (2.1) *is stable if for any given* $t_0 \in E$*, and a positive number* ε *there exists* $\delta = \delta(to, \varepsilon)$ *such that* $\phi \in \beta(0, \delta)$ *implies* $x_t(t_0, \phi) \in \beta(0,\varepsilon)$ *for all* $t > t_0$. $\phi \in C[-h,0]$ and $\beta(o,r)$, *a ball centre at* 0 *with radius r*.

***Definition* 3.2**

The trivial solution $x = 0$ *of* (2.1) *is uniformly stable if for any given* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon)$ (*independent of* t_0) *such that*

$$\phi \in \beta(0,\delta) \text{ implies } x_t(t_0,\phi) \in \beta(0,\varepsilon) \text{ for all } t > t_0.$$

Definition 3.3

The trivial solution $x = 0$ of (3.1) is asymptotically stable, if it is stable, such that $\phi \in \beta(0, \delta)$ implies.

$$x_t(t_0,\phi) \to 0 \text{ as } t \to \infty \qquad (3.1)$$

In the case where the trivial solution is uniformly stable and satisfies condition (3.1) the system is said to be uniformly asymptotically stable.

***Definition* 3.4**

The solution $x(t_0, \phi)$ of (3.1) is exponentially asymptotically stable in the large if there exist $L > 0$ and $c > 0$ such that the solution satisfies

$$x_t(t_0,\phi) = \phi \text{ and } \|x_t(t_0,\phi)\| \le Le^{-C(t-to)}\|\phi\|.$$

***Lemma* 3.1**

Let $V(t)$ *and* $P(t)$ be *continuous functions for* $t \ge t_0$. *Let* $C \ge 0$, $M \ge 0$ and $V(t) \le C + \int_0^t \{MV(s) + P(s)\}ds$ then $V(t) \le Ce^{M(t-t_0)} + \int_0^t P(s)e^{M(t-s)}\,ds$

Proof

See Chukwu [5]

***Lemma* 3.2**

Suppose $\int_0^{t+1} \pi(s)ds \to 0$ as $t \to \infty$, *then* $\lim_{t\to\infty} e^{-\alpha t}\int_1^t e^{\alpha s}\pi(s)ds = 0$ for all $t \succ 1$.

Proof

See Chukwu [5]

4.0 Main Results

4.1 Stability criteria for linear systems with unbounded delays

A stability result for linear delay systems is stated and proved via the exponential estimate of the solutions. This result complements known results (see Chukwu [5], Hale [10]). To motivate our discussion on exponential asymptotic stability, we consider exponential estimate of the solution of system (2.1).

$$x(t) = L(t, x_t).$$

By direct integration, we have $x(t) = \int_{t_0}^t L(s, x_t)ds + C$

Exploiting the boundedness condition on L, i.e. $||L(t,x_t)|| \le K|x_t||$

we have
$$x(t) = \int_{t_0}^t K\|x_s\|ds + x(t_0) \qquad (4.1)$$

Clearly $x(t_0) = C$. by the initial condition. It is know that for all values of
$$x, x < e^x \text{ (see [15])} \qquad (4.2)$$

Apply (42), into (4.1) to have $\|x(t)\| \prec e^{\|x(t)\|} \prec e^{(\|x(t_0)\|)} + k\int_{t_0}^t \|x_s\|ds$,

$$\|x(t)\| \le e^{\|x(t_0)\|}e^k\int_{t_0}^t \|x_s\|ds \le M\exp\left(\int_{t_0}^t \|x_s\|\, ds\right), \text{ M is a positive constant} \qquad (4.3)$$

This Gronwall type integral inequality is the required exponential estimate of the solution of system (2.1)

***Proposition* 4.1**

System 2.1 with its basic assumptions is exponentially stable if the exponential estimate tends to zero as $t \to \infty$

Proof

If k is negative in 3.3, we observe that the solution of (2.1) tends to zero as $t \to \infty$ faster that any exponential function. In which case the system (2.1) is said to be exponentially asymptotically stable. If this condition holds for any initial value of the state vector, the system is said to be exponentially asymptotically stable in the large or globally exponentially, asymptotically stable.

If the exponential estimate of the solution ordinarily approaches zero as $t \to \infty$ then system (2.1) is said to be asymptotically stable. Evidently, if system (2.1) is exponentially, asymptotically stable then it is asymptotically stable. However, the converse does not necessarily hold.

We here provide an example to illustrate the concept of exponential asymptotic stability.

Consider the system

$x(t) = -(1 + \sin^2 x(t))\, x(t)$ with the initial condition $x(t_0) = x_0$, By direct integration, we have

$$\ln x(t) = -\int_{t_0}^t (1+\sin^2 x(s))ds + C$$
$$x(t) = \exp\left(-\int_{t_0}^t (1+\sin^2 x(x))ds + C\right) = x_0\exp - \int_{t_0}^t (1+\sin^2 x(s))ds$$

The condition $||x(t)|| \to 0$ as $t \to \infty$ holds, showing that the trivial solution of the system becomes
$$||x(t)|| \le ||x(t_0)||e^{-t}$$

Thus establishing exponential asymptotic stability. The solution, of course approaches zero faster than any exponential function. Since the result holds for any initial value of the state, we have exponential asymptotic stability in the large.

Evidently, exponential asymptotic stability implies uniform asymptotic stability, however the converse is not necessarily true.

4.2 Stability criteria for linear perturbations

We shall now state and prove the necessary and sufficient conditions for perturbations of linear systems with unbounded delays to be exponentially, asymptotically stable in the large.

***Theorem* 4.1**

Consider system (2.2)

$$\dot{x}(t) = L(t, x_t) + f(t, x_t) \tag{4.4}$$

where $L(t, x_t) = \sum_{k=0}^{\infty} A_k x(t - wk) + \int_{-\infty}^{0} A(t, \theta) x(t + \theta)$ *with its basic assumptions, Suppose* (i) *system* (2.1) *is uniformly asymptotically stable* (ii) *the function* $f = f_1 + f_2$ *satisfies the condition* $f_1(t, \phi) \leq \pi(t)||\phi||$; $t > 0$, $\phi \varepsilon C$ *where* $\pi = \int_{t_0}^{t} \pi(t) dt \prec \infty$ *and there exists an* $\varepsilon > 0$ *such that* $|f_2(t, \phi)| \leq \varepsilon \|\phi\|$; $t \succ 0$, $\phi \notin C$. *Assume further that there exists a continuous Lyapunov functional* $V(t, \phi)$ *defined on* $[o, \infty]$ x C *which satisfies the following conditions.*

(i) $||\phi|| < V(t, \phi) \leq M||\phi||$

(ii) $|V(t, \phi) - V(t, \psi)| \leq M||\phi - \psi||\phi, \psi \varepsilon C$

(iii) $V(t, \phi) < -\alpha V(t, \phi)$

then system (2.1) *is exponentially asymptotically stable in the large*.

Proof

From condition (ii), $V(t, \phi) \leq M$ for $\phi \varepsilon C$. Condition (iii) and the hypothesis on *f* show that $V(t, \phi) \leq -\alpha V(t, \phi) + M|F(t, \phi)|$ for all $t > t_o$, $\phi \varepsilon C$. We now show that system (2.2) is exponentially asymptotically stable in the large, that is there exist an $L > 0$, $c > 0$ such that the solution $x_t(t_o, \phi)$ of system (2.2) satisfies $x_t(t_o, \phi) = \phi$ and $||x_t(t_0, \phi)|| < Le^{-c(t - to)}||\phi||$. Since the linear system (2.1) is assumed uniformly asymptotically stable, from Hale [10], there exist constants $M > 0$, $\alpha > 0$ such that for every $t > t_0$ and $\phi \varepsilon C$ the solution $x(t_0, \phi)$ of (2.1) satisfies $||x_t(t_0, \phi)|| \leq Me^{-\alpha(t - to)}||\phi||$; $t \geq t_o$. Let us consider a function $Z(t, \phi)$ such that $Z(t, \phi) = V(t, \phi) \exp\left\{-M \int_{t_0}^{t} \pi(s)\, ds\right\}$ which is defined on I x C. Choose $\varepsilon = \frac{\alpha}{2M}$. In this domain,

$$Z(t, \phi) = \exp\left\{-M \int_{t_0}^{t} \pi(s) ds\right\}\left\{V - M\pi(t)V\right\} \leq \exp\left\{-M \int_{t_0}^{t} \pi(s) ds\left\{-M\pi(t)V(t, \phi) - \alpha V(t, \phi) + M\left(|F_1(t, \phi)| + F_2|(t, \phi)|\right)\right\}\right.$$

$$\leq \exp\left\{-M \int_{t_0}^{t} \pi(s) ds\right\}\left\{-M\pi(t)V(t, \phi) - \alpha V(t, \phi) + M\pi(t)V(t, \phi) + M\varepsilon V(t, \phi)\right.$$

Hence

$$\leq (M\varepsilon - \alpha)V(t, \phi) \exp\left\{-M \int_{t_0}^{t} \pi(s) ds\right\} \leq -\frac{\alpha}{2} Z(t, \phi).$$

$Z(t, x_t(t_o, \phi)) \leq Z(t_o, \phi)e^{(-\alpha/2)(t - t_o)}$

This implies

$$\|x_t(t_0, \phi)\| \leq M\|\phi\| \exp\left\{M \int_{t_0}^{t} \pi(t) ds\right\} \exp\left(-\frac{\alpha}{2}(t - t_0)\right)$$

with $L = M \exp\left(M \int \pi(t) dt,\ c = \frac{\alpha}{2}\right)$

We establish the required result.

The next theorem further shows that a strong relationship exists between exponential asymptotic stability and uniform asymptotic stability.

***Theorem* 4.2**

Suppose that (i) system (2.1) *is uniformly asymptotically stable so that for some* $k \geq 1$, $a > 0$ the *solution* $xt(t_o, \phi)$ of (2.1) *satisfies* $||x_t(t_0, \phi)|| \leq ke^{-\alpha(t - to)}||\phi||$; $t > t_o$, $\phi \varepsilon C$.

(ii) *the function* $f = f_1 + f_2$ *satisfies the condition*

$$|f_1, (t, \phi)| \leq \pi(t);\ t \geq,\ \phi \in C$$

where $\Pi(t) = \int_{t}^{t+1} \kappa(s) ds \to 0$ as $t \to \infty$

$$|f_2, (t, \phi)| \leq \kappa\|\phi\|;\ t \succ \phi \in C;\ \varepsilon = \frac{\alpha}{2k}$$

Then every solution $x_t(t_0, \phi)$ *of* (2.1) *is uniformly asymptotically stable*.

Proof

By the variation of constant formula, the solution of (2.2) can be expressed by

$$x(t_0,\phi,f)=x_t(t_0,\phi,0)+\int_{t_0}^{t} Xt_0(\theta,s)dsf(s,x_s)$$

where $x_t(t_0, \phi, 0)$ is the solution of system (2.1) with

$$x_{t0}(t_0, \phi) = \phi \text{ and } X_t(\theta,s) = X(t+\theta, s),\ \theta\ \varepsilon\ [-h, 0]$$

where $X(t, s)$ is the fundamental matrix for system (2.1). By uniform asymptotic stability of (2.1) as spelt out in Hale [10],

$$||X(t, s)|| < ke^{-\alpha(t-s)};\ t \geq s$$

holds. We now obtain an estimate of the solution $xt(t_0, \phi)$ of (2.2)

$$\|x_t(t_0,\phi)\| \le k\|\phi\|e^{-\alpha(t=s)}+\int_{t_0}^{t}\varepsilon ke^{-\alpha(t=s)}\|x_s\|ds+\int_{t_0}^{t}ke^{-\alpha(t=s)}\kappa(s)ds,\ t\ge t_0$$

so that $$\|x_t\|e^{at} \le k\|\phi\|e^{-\alpha(t_0)}+\int_{t_0}^{t}\{\varepsilon ke^{\alpha s}\|x^s\|+ke^{\alpha s}\kappa(s)\}ds$$

By the lemma (2.1), If we apply $V(t) = ||x_t||e^{\alpha t}$ then

$$\|x_t\|e^{at} \le k\|\phi\|e^{-\alpha t_0}e^{k\varepsilon(t-t_0)}+\int_{t_0}^{t}ke^{\alpha s}\kappa(s)e^{k\varepsilon(t-s)}ds$$

so that $$\|x_t\| \le k\|\phi\|e^{-(\alpha-k\varepsilon)(t-t_0)}+k\int_{t_0}^{t}e^{-(\alpha-k\varepsilon)(t-s)}\kappa(s)ds;\ t\ge t_0$$

or $$\|x_t\| \le k\|\phi\|e^{-\left(\frac{\alpha}{2}\right)(t-t_0)}+k\int_{t_0}^{t}e^{-\left(\frac{\alpha}{2}\right)(t-s)}\kappa(s)ds;$$

Therefore $$\|x_t\| \le k\|\phi\|e^{-\left(\frac{\alpha}{2}\right)(t-t_0)}+k\int_{1}^{t}e^{-\left(\frac{\alpha}{2}\right)(t-s)}\kappa(s)ds;$$

By Lemma (2.2), if π is given as in condition (ii) of the lemma then

$$\lim_{t\to\infty} e^{-\alpha s}\int_1^t e^{\alpha s}\kappa(s)ds=0;\quad \text{for all } t\succ 1$$

It follows that $\|x_t(t_0,\phi)\| \to 0,$ as $t\to\infty$

This concludes the proof

To illustrate theorems (4.1) and (4.2), we consider the follow example.

***Example* 4.1**

Consider the system $$\dot{x}(t)=Ax(t)+Bx(t-1)+F(t,x_t) \tag{4.5}$$

where $A=\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix},\ B=\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}$

$f(t, x_t) = [e^{-t}(\log \sin x(t) + \cos x(t-1))x(t-1)$.

To investigate the asymptotic behaviour of the solution of the system, we consider first, the linear part of system (4.5) $$\dot{x}(t)=Ax(t)+Bx(t-1) \tag{4.6}$$

Its characteristic equitation is obtained thus

$$\Delta(\lambda)=(\lambda 1-(A+Be^{-\lambda}))$$

$$=\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}-\left\{\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & -e^{-\lambda}\end{pmatrix}\right\}\right\}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}-\begin{pmatrix}0 & 1\\ -2 & -3-e^{-\lambda}\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & \lambda+3-e^{-\lambda}\end{pmatrix}$$

$$\det\Delta(\lambda)=(\lambda^2+3\lambda+\lambda e^{-\lambda}+2=0)$$

Comparing this result with the equation $\lambda^2 + b\lambda + q\lambda e^{-\lambda h} + k = 0$ which according to Driver [6] in pp. 32, example 4.1 will have negative real parts for its roots if $b > q$, $b > 0$, $q > 0$. In the light of this result, we conclude that the roots of the characteristic equation for system (4.6) have negative real parts and hence the system is uniformly asymptotically sable by Theorem (4.4)

Also observe that

$$|f(t, x(t), x(t-1))| = [e^{-t}(\log \sin x(t) + \cos x(t-1))x(t-1)]$$

$\left|[e^{-t}(\log \sin x(t))x(t-1)]\right| + e^{-t}\left|[\cos x(t-1))x(t-1)]\right|$

$\left|f_1(t, \phi)\right| = e^{-t}.0.(t-1) = 0, \left|f_2(t,\phi)\right| = \left|e^{-t}x(t-1)\right| = e^{-t}\left|\left|\phi\right|\right|$

Let $\kappa(t) = e^{-t}$ then $\int_t^{t+1} e^{-s}ds \to 0$ as $t \to \infty$, clearly, $\left|f_1(t, \phi)\right| < \pi(t)$ and $f_2(t, \phi) < \varepsilon\left|\left|\phi\right|\right|$, $t \geq 1$, $\phi\ \varepsilon$ C, f_1 and f_2 are bounded and by theorem (4.1), we conclude that system (2.2) is exponentially asymptotically stable in the large; and by theorem (4.2), the system is uniformly asymptotically stable.

From the above examples, it is not just sufficient to know that a system is uniformly asymptotically stable. There is need to investigate how fast the state of the system approaches zero and this establishes the significance of the concept of exponential asymptotic stability in the large.

5.0 Conclusion

The equation studied is a perturbation of linear delay system with infinite and unbounded delays in the state. Necessary and sufficient conditions are proved for the asymptotic stability and the exponential asymptotic stability in the large of the system (2.2) under reference. The distinction between the two variants of asymptotic stability are made. Asymptotic stability guarantees that the solution, $|x(t)|$ of system (2.2) approaches zero as $t \to \infty$ while exponential asymptotic stability in the large is concerned with the rate at which the solution $|x(t)|$ of the system approaches zero . This is inseparable from the exponential boundedness of the solution, which is mathematically expressed as $\|x(t)\| \leq k\|\phi\|e^{c(t-t_0)}$ for $t > t_o$ and c, k constants.

The Lyapunov function and theorem played key role in establishing our results. We exploited the condition that the derivative of the Lyapunov function must be negative definite and have small upper bound to guarantee asymptotic stability. We illustrated our results using illuminating examples. This research is an extension of the work in [5] and [7], carrying over these previous results to systems with infinite and unbounded delays. A new method of handling equations of this type presents a fascination as it exploits the convergence of sum of series and improper integrals to achieve the desired results.

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