Journal of the Nigerian Association of Mathematical Physics
Volume 10 (November 2006), **523 - 528**
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Relative controllability of nonlinear systems with multiple delays in state and control

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Abstract

Sufficient conditions are developed for the relative controllability of nonlinear systems with time-varying multiple delays in the state and control. The results are obtained by defining an appropriate control and its corresponding solution by an integral equation. This equation is then solved using the Schauder's fixed point theorem.

1.0 Introduction.

This paper is mainly concerned with the relative controllability on a bounded interval $[0,t_1]$ of nonlinear systems with time varying multiple delays in state and control of the form

$$\dot{x}(t)=\sum_{i=0}^{p}A_i(t)x(t-h_i)+\sum_{i=0}^{p}B_i(t)u(t-h_i)+f(t,x(t),x(t-h),u(t),u(t-h)),t\in[0,t_1]$$

$$x(t)=\phi(t),\ t\in[-h,0] \tag{1.1}$$

Mathematical models of this type play an important role in every filed of science where causes do not produce their effects immediately but with some time delay. Systems with delayed control are natural models for the study of some economic, biological and physiological systems as well as electromagnetic systems composed of subsystems interconnected by hydraulic and various other linkages. For such systems, the decisions in the control function are often shifted, twisted or combined before affecting the evolution [13]. Controllability problems for different types of nonlinear dynamical systems with delays in control have been studied by several authors with the help of different fixed point theorems [1 – 12]. Decka [4] used the notion of measure of noncompactness of a set and Darbo's fixed point theorem to establish the relative controllability for nonlinear systems with time varying multiple delays in control and with implicit derivative. Klamka [7, 8] used the notion of linearization and Schauder's fixed point theorem to study the relative controllability of nonlinear systems with distributed delays in the control. However, for nonlinear systems with time varying multiple delays in both the state and control variables, the problem of relative controllability is still to be studied. This constitutes the main objective of our research.

It is now well known that true life dynamical systems are often nonlinear since they are often affected by frictions, hitches, noises, etc. It is therefore expedient that the conditions under which such systems are controlled be investigated. We shall show that if the linear delay system

$$\dot{x}(t)=\sum_{i=0}^{p}A_i(t)x(t-h_i)+\sum_{i=0}^{p}B_i(t)u(t-h_i) \tag{1.2}$$

is relatively controllable, then the perturbed system (1.1) is relatively controllable provided the perturbation function f satisfies appropriate growth conditions.

2.0 Preliminaries

In equation (1.1), $x(t)$ is an n-vector, $u(t)$ is measurable continuous m-vector function; A_i are $n\times n$ continuous matrices, B_i are $n\times m$ continuous matrices and $\phi(t)$ is a continuous vector function on the interval $[-h,0]$. The control function $u(t)\in E^m$ is assumed to be measurable and bounded on every finite interval. Here $E=(-\infty,\infty)$, the real line and E^n is the n-dimensional Euclidean space with norm, $|.|$. We let $C=C([-h,0],E^n)$ be the Banach space of continuous functions and we designate the norm of an element ϕ in C by $\|\phi\|=\sup_{-h\le s\le 0}|\phi(s)|$. We let $L_1([a,b],E^m)$ be the space of Lebesgue integrable functions taking $[a,b]$ into E^n with $\|\phi\|=\int_a^b|\phi(s)|ds, \phi\in L_1([a,b],E^n)$. $L_\infty([a,b],E^m)$ is the space of essentially bounded functions taking $[a,b]\to E^n$ with $\|\phi\|=es\sup_{s\in[a,b]}|\phi(s)|$. If $X\in C([a,b],E^m)$ for any $a\le b$, then for each fixed $t\in[a,b]$, the symbol x_t denotes an element of C given by $x_t(s)=x(t+s)$, $h\le s\le 0$. Similarly, for functions $u\in L_\infty([a,b],E^m$, the symbol u_t denotes an element of L_∞ given by $u_t(s)=u(t+s), -h\le s\le 0$. Throughout the sequel, the controls of interest are:

a) $IB=L_\infty([0,t_1],E^m)$

b) $IU=L_\infty([0,t_1],C^m)$

where $C^m=\{u:u\in E^m, |u_j|\le 1, j=1,2...,m\}$.

The above conditions on A_i and B_i ensure that for each initial data $(0,\phi)$, a unique solution of system (1.1) exists through $(0,\phi)$ (see Hale [6, pp 143]) which is continuous in $(0,\phi)$. The solution of (1.1) is given by

$$x(t_1)=X(t_1,0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)A_i(t+h_i)\phi(t)dt+\int_0^{t_1}X(t_1,t)\sum_{i=0}^{p}B_i(t)u(t-h_i)dt$$
$$+\int_0^{t_1}X(t_1,t)f(t,x(t),x(t-h),u(t),u(t-h))dt \qquad (2.1)$$

This formula can be rewritten as

$$x(t_1)=X(t_1,0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)A_i(t+h_i)\phi(t)dt+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)B_i(t+h_i)u_0(t)dt$$
$$+\sum_{i=0}^{p}\int_0^{t_1-h_i}X(t_1,t+h_i)B_i(t+h_i)u(t)dt+\int_0^{t_1}X(t_1,t)f(t,x(t),x(t-h),u(t),u(t-h))dt \qquad (2.2)$$

where $X(t,s)$ is the fundamental matrix solution of the homogeneous part of (1.1) which satisfies the equations

$$\frac{\delta}{\delta t}X(t,s)=\sum_{i=0}^{p}A_i(t)X(t-h_i,s), t>s$$

$$X(t,s)=\begin{cases} I, & t=s \\ 0 & t<s \end{cases}$$

or $$\frac{\delta}{\delta s}X(t,s)=-\sum_{i=0}^{p}X(t,s+h_i)A_i(s+h_i), t>s.$$

Define $Z(t_1,t)=\sum_{i=0}^{p}X(t_1,t+h_i)B_i(t+h_i)$ and the controllability matrix $W(0,t_1)=\int_0^{t_1}Z(t_1,t)Z^T(t_1,t)dt$ where T denotes the matrix transpose.

***Definition* 2.1**

The set $Z(t)=\{x(t),x_t,u_t\}$ is said to be the complete state of the system (1.1) at time t.

***Definition* 2.2**

The system (1.1) is said to be relatively controllable on $[0,t_1]$ if for every initial complete state

$Z(0)=\{x(0),\phi,u_0\}$ and every vector $x_1\in E^n$, there exists a control $u\in IB$ such that the solution of the system (1.1) satisfies $x(t_1)=x_1$.

3.0 Main results

***Theorem* 3.1**

In (1.1) *assume*;

i) *system* (1.2) *is relatively controllable on* $[0,t_1]$

ii) *the continuous function* f *satisfies the growth condition*

$$\lim_{\|(x,u)\|\to\infty}\frac{|f(t,x,u)|}{\|x,u\|}=0$$

uniformly in $t\in[0,t_1]$, then system (1.1) is relatively controllable on $[0,t_1]$.

Proof

From equation (2.2), the solution of system (1.1) can be rewritten as

$$x(t_1)=x_L(t_1)+\int_0^{t_1}X(t_1,t)f(t,x(t),x(t-h),u(t),u(t-h))dt$$

where $$x_L(t_1)=x_L(t_1,\phi)+\sum_{i=0}^{p}\int_o^{t_1-h_i}X(t_1,t+h_i)B_i(t+h_i)u(t)dt$$

and $$x_L(t_1,\phi)=x_L(t_1,0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)A_i(t+h_i)\phi(t)dt+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)B_i(t+h_i)u_0(t)dt$$

Let Q be the Banach space of all functions $(x,u):[0,t_1]\times[-h,t_1]\to E^n\times E^m$, where x is continuous and u is an admissible control function. The norm on Q is

$$\|(x,u)\|=\|x\|+\|u\|$$

where

$$\|x\|=\sup|x(t)| \text{ for } t\in[0,t_1]$$

$$\|u\|=\sup|u(t)| \text{ for } t\in[-h,t_1]$$

Let $T:Q\to Q$ be an operator defined by

$$T(x,u)=(y,v)$$

where $v(t)=Z^T(t_1,t)W^{-1}[x_1-x_L(t_1,\phi)-\int_0^{t_1}X(t_1,t)f(t,x(t),x(t-h),u(t),u(t-h))dt]$ for $t\in[0,t_1]$, and $v(t)=0$ for $t\in[-h,0],x_1\in E^n$;

$y(t)=x_L(t_1,\phi)+\int_0^{t_1-h_i}Z(t_1,t)v(t)dt+\int_0^{t_1}X(t_1,t)f(t,x(t).x(t-h),u(t),u(t-h))dt$ for $t\in[0,t_1]$, and $y(t)=\phi(t)$ for $t\in[-h,0]$.

Observe that the control $v(t)$ is capable of steering the solution of system (1.1) to x_1 at $t=t_1$.

Let $a_1=\sup|Z(t_1,t|$ for $0\le t\le t_1$

$a_2=|W^{-1}|$

$a_3=\sup|x_L(t_1,\phi)|+|x_1|$ for $t\in[-h,0]$

$a_4=\sup|X(t_1,t)|$ for $(t_1,t)\in[0,t_1]\times[0,t_1]$

$b=\max\{t_1,a_1,h\}$

$c_1=8ba_1a_2a_4t_1$

$c_2=8a_4t_1$

$d_1=8a_1a_2a_3b$

$d_2=8a_3$

$c=\max\{c_1,c_2\}$

$d=\max\{d_1,d_2\}$.

Then $\quad |v(t)| \le a_1 a_2 [a_3 + a_4 t_1 (\sup |f(t, x(t), x(t-h), u(t), u(t-h))| \text{ for } t \in [0, t_1])]$

$$= \frac{d_1}{8b} + \frac{c_1}{8b}(\sup |f(t, x(t), x(t-h), u(t), u(t-h))| \text{ for } t \in [0, t_1]$$

$$\le \frac{1}{8b}[d + c(\sup |f(t, x(t), x(t-h), u(t), u(t-h))| \text{ for } t \in [0, t_1])];$$

$$|y(t)| \le a_3 + a_1 \|v\| t_1 + t_1 a_4 (\sup |f(t, x(t), x(t-h), u(t), u(t-h))| \text{ for } t \in [0, t_1])$$

$$\le b\|v\| + \frac{d}{8} + \frac{c}{8}(\sup |f(t, x(t), x(t-h), u(t), u(t-h))| \text{ for } t \in [0, t_1]).$$

Let f satisfy the following condition: for each pair of positive constants c and d, there exists a positive constant r such that, if $\|x, u\| \le r$, then $c|f(t, x, u)| + d \le r$ for all $t \in [0, t_1]$. Let r be chosen so that this implication is satisfied and

$$\sup_{-h \le t \le 0} |\phi(t)| \le \frac{r}{4}.$$

Therefore, if $\quad \|x\| \le \frac{r}{4}$ and $\|u\| \le \frac{r}{4}$ then

$$|x(t)| + |x(t-h)| + |u(t)| + |u(t-h)| \le r \text{ for all } t \in [0, t_1].$$

It follows that

$$d + c(\sup |f(t, x(t), x(t-h), u(t), u(t-h))| \text{ for } t \in [0, t_1] \le r.$$

Therefore,

$$|v(t)| \le \frac{r}{8b} \text{ for all } t \in [0, t_1]$$

and hence

$$\|v\| \le \frac{r}{8b}.$$

It follows that

$$|y(t)| \le \frac{r}{8} + \frac{r}{8} \text{ for all } t \in [0, t_1]$$

and hence

$$\|y\| \le \frac{r}{4}.$$

We have just shown that if

$$Q(r) = \left\{(x, u) \in Q : \|x\| \le \frac{r}{4} \text{ and } \|u\| \le \frac{r}{4}\right\}$$

then

$$T : Q(r) \to Q(r)$$

where $Q(r)$ is a bounded set, hence T is well defined.

We shall now show that T is completely continuous or T is a sequentially compact operator. Since f is continuous, it follows that T is continuous. Let Q' be a bounded subset of Q. Consider the sequence $\{(y_i, v_j)\} \in T(Q'),$, such that $(y_i, v_j) = T(x_j, u_j)$ for some $(x_j, u_j) \in Q'$ for $j = 1, 2....$ Since f is continuous, $|f(t, x(t), x(t-h), u(t)u(t-h))|$ is uniformly bounded for all $t \in [0, t_1]$. It follows that $\{y_i, v_j\}$ is a bounded sequence in Q. Hence $\{v_j(t)\}$ is an equicontinuous and a uniformly bounded sequence on $[-h, t_1]$. Since each $v_j(t)$ has both right and left hand limits at $t = 0$ and $t = t_1 - h$, we can apply Ascoli's theorem on $[0, t_1 - h]$ to the sequence $\{v_j(t)\}$. Therefore, there exists a subsequence of $\{v_j(t)\}$ which converges uniformly to a continuous function on $[0, t_1 - h_i]$. Also, since $\{y_j(t)\}$ is a uniformly bounded and equicontinuous sequence on $[-h, t_1]$, a further application of Ascoli's theorem yields a further subsequence $\{(y_i, v_j)\}$ which converges in Q to some

(y_0, u_0). It follows that $T(Q')$ is sequentially compact. Hence the closure $\{T(Q')\}$ is sequentially compact. Thus, T is completely continuous. Since $Q(r)$ is closed, bounded and convex, the Schauder fixed-point theorem implies that T has a fixed point $(x,u) \in Q(r)$ which is the required solution of system (1.1) capable of satisfying the boundary conditions $x(0) = x_0$ and $x(t_1) = x_1$ for $t_1 > 0$ and $x_0, x_1 \in E^n$.

This by implication means that

$$x(t_1) = x_L(t_1, \phi) + \int_0^{t_1 - h_i} Z(t_1, t)u(t)dt + \int_0^{t_1} X(t_1, t) f(t, x(t), x(t-h), u(t), u(t-h))dt$$

for $t \in [0, t_1]$ and $x(t) = \phi(t)$ for $t \in [-h, 0]$. Hence, $x(t)$ is a solution of system (1.1) and

$$x(t_1) = x_L(t_1, \phi) + \int_0^{t_1 - h_i} Z(t_1, t) Z^T(t_1, t) W^{-1}[x_1 - x_L(t_1, \phi) - \int_0^{t_1} X(t_1, t) f(t, x(t), x(t-h), u(t), u(t-h))dt]dt$$

$$+ \int_0^{t_1} X(t_1, t) f(t, x(t), x(t-h), u(t), u(t-h))dt = x_1$$

Hence, system (1.1) is relatively controllable on $[0, t_1]$.

4.0 Conclusion

Using Schauder's fixed point theorem sufficient conditions for relative controllability in a given finite time interval for nonlinear dynamical systems with time varying multiple delays in state and control have been derived. The results obtained extend known results to nonlinear systems with multiple delays in both the state and control variables.

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