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Relative null controllability of linear systems with multiple delays in state and control

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Abstract

Sufficient conditions for the relative null controllability of linear systems with time–varying multiple delays in state and control are developed. If the uncontrolled system is uniformly asymptotically stable, and if the linear system is controllable, then the linear system is null controllable.

Key words**:** Null controllability, multiple delays, time-varying.

1.0 Introduction.

The problem of null controllability of linear systems has been extensively studied by several authors [7 – 9]. It is reported in [4] that if the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.1}$$

is proper and if

$$\dot{x}(t) = A(t)x(t) \tag{1.2}$$

is uniformly asymptotically stable then (1.1) is null controllable.
Chukwu [1] showed that if the linear delay system

$$\dot{x}(t) = L(t, x_t) \tag{1.3}$$

is uniformly asymptotically stable and

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) \tag{1.4}$$

is proper, then

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) + f(t, x_t, u(t)) \tag{1.5}$$

is Euclidean null controllable provided f satisfies certain growth and continuity conditions. He also showed in [2] that if (1.3) is uniformly asymptotically stable, and (1.4) is function space controllable, then (1.5) is function space null controllable with constraints.
Chukwu [5] studied the null controllability of systems of the form

$$\dot{x}(t) = L(t, x_t) + \sum_{i=0}^{p} B_i(t)u(h_i(t)) \tag{1.6}$$

with constant and distributed delays in control with respect to the control of global economic growth. Further, Chukwu introduced the solidarity functions in (1.6) and obtained certain universal principles for the control of economic growth of interconnected systems.
On optimality, Onwuatu [3] studied the system

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{p} B_j x(t-j) + \sum_{j=0}^{p} D_j u(t-j) \tag{1.7}$$

and gave sufficient conditions for the optimal control of (1.7). In this paper, linear systems with time–varying multiple delays of the following form are considered:

$$\dot{x}(t) = \sum_{i=0}^{p} A_i(t)x(t-h_i) + \sum_{i=0}^{p} B_i(t)u(t-h_i), \; t \in [0, t_1] \tag{1.8}$$

$$x(t) = \phi(t), \; t \in [-h, 0]$$

This is motivated by the fact that time delay is frequently encountered in various engineering, communication, and biological systems. The characteristics of dynamical systems are significantly affected by the presence of time delays, even to the extent of instability in extreme situations.

Owing to the obvious difficulty in handling the many time lags in both the state and control variables, not many studies are undertaken to establish sufficient conditions for the controllability as well as the null controllability of linear systems with time varying multiple delays in state and control. The present endeavour is to establish such conditions. This is the thrust of our research.

2.0 Preliminaries

In equation (1.8), $x(t)$ is an n-vector, $u(t)$ is a measurable m-vector continuous function; A_i are $n\times n$ continuous matrices, B_i are $n\times n$ continuous matrices and $\phi(t)$ is a continuous vector function on the interval $[-h,0]$. The control function $u(t)\in E^m$ is assumed to be measurable and bounded on every finite interval. Here $E=(-\infty,\infty)$, the real line and E^n is the n–dimensional Euclidean space with norm, $|.|$. We let $C=C([-h,0],E^n)$ be the Banach space of continuous functions and we designate the norm of an element ϕ in C by $\|\phi\|=Sup_{-h\le s\le 0}|\phi(s)|$. We let $L_1([a,b],E^m)$ be the space of Lebesgue integrable functions taking $[a,b]$ into E^n with $\|\phi\|=\int_a^b|\phi(s)|\,ds,\quad \phi\in L_1([a,b],E^n)$. $L_\infty([a,b],E^m)$ is the space of essentially bounded functions taking $[a,b]\rightarrow E^n$ with $\|\phi\|=es\sup_{s\in[a,b]}|\phi(s)|$. If $x\in C([a,b],E^n)$, for $a\le b$ then for each fixed $t\in[a,b]$, the symbol x_t denotes an element of C given by $x_t(s)=x(t+s)$, $-h\le s\le 0$. The function u_t is similarly defined.

The above conditions on A_i and B_i ensure that for each initial data $(0,\phi)$, a unique solution of (1.8) exits through $(0,\phi)$ (see Hale [6, pp. 143]) which is continuous in $(0,\phi)$. The solution of (1.8) is given by

$$x(t_1,0,\phi,u)=X(t_10)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)A_i(t+h_i)\phi(t)dt+\int_0^{t_1}X(t_1,t)\sum_{i=0}^{p}B_i(t)u(t-h_i)dt \qquad (2.1)$$

This formula can be rewritten as

$$x(t_1,0,\phi,u)=X(t_1,0)\phi(0)+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)A_i(t+h_i)\phi(t)dt$$
$$+\sum_{i=0}^{p}\int_{-h_i}^{0}X(t_1,t+h_i)B_i(t+h_i)u_0(t)dt+\sum_{i=0}^{p}\int_0^{t_1-h_i}X(t_1,t+h_i)B_i(t+h_i)u(t)dt \qquad (2.2)$$

where $X(t,s)$ is the fundamental matrix solution of the homogeneous part of (1.8) which satisfies the equation $\frac{\partial}{\partial t}\mathrm{X}(t,s)=\sum_{i=0}^{p}\mathrm{A}_i(t)\mathrm{X}(t-h_i,s),\ \mathrm{t}>\mathrm{s}$.

$$X(t,s)=\begin{cases}\mathrm{I}, & \mathrm{t}=\mathrm{s}\\ 0, & \mathrm{t}<\mathrm{s}\end{cases}\quad \text{or}\quad \frac{\partial}{\partial s}X(t,s)=-\sum_{i=0}^{p}X(t,s+h_i)A_i(s+h_i),t>s$$

Define $Y(t_1,t)=\sum_{i=0}^{p}X(t_1,t+h_i)B_i(t+h_i)$ and the controllability matrix $W(0,t_1)=\int_0^{t_1}\mathrm{Y}(t_i,t)\mathrm{Y}^T(t_1,t)dt$ where T denotes matrix transpose.

***Definition* 2.1**

The system (1.8) *is said to be relatively null controllable on* $[0,t_1]$ *if for each* $\phi\in C([-h,0],E^n)$, *there exist a* $t_1>0$, *and a* $u\in L_\infty([0,t_1],E^m)$ *such that the solution* $x(t_1,0,\phi,u)$ *of* (1.8) *satisfies* $x(t_1,0,\phi,u)=\phi$ *and* $x(t_1,0,\phi,u)=0$.

***Definition* 2.2**

The reachable set $R(t_1,0)$ *of* (1.8) *at time* t_1 *is a subset of* E^n *given by*

$$R(t_1,0)=\left\{\int_0^{t_1} Y(t_1,t)u(t)dt : u\in L_\infty\left([0,t_1],E^m\right)\right\}$$

In a similar manner, we define the constraint reachable set of (1.8) *at time* t_1 *as*

$$IR(t_1,0)=\left\{\int_0^{t_1} Y(t_1,t)u(t)dt : u\in L_\infty\left([0,t_1],E^m\right)\right\}$$

where $C^m=\left\{u : u\in E^m, \left|u_j\right|\le 1,\ j=1,2,\ldots,m\right\}$.

***Definition* 2.3**

The system (1.8) *is said to be proper in* E^n *on* $[t_0,t_1]$ if $c^T[Y(t_1,t)]=0\ a.e. t\in[t_0,t_1], c\in E^n$ implies *that* $c=0$, *where* c^T *is the transpose of* c .

***Definition* 2.4**

The domain D *of relative null controllability of system* (1.8) *is the set of all initial functions* $\phi\in C$ *which can be steered to the origin* $0\in E^n$ *in finite time, using admissible controls. From the above developments we now proceed to establish in this section the crucial facts leading to the main result of this paper. Firstly, we have the following:*

***Theorem* 2.1**

The following are equivalent:

(i) $W(0,t_1)$ *is nonsingular for each* $t_1>0$;

(ii) *system* (1.8) *is proper in* E^n *for each interval* $[t_0,t_1]$;

(iii) *system* (1.8) *is relatively controllable on each interval* $[t_0,t_1]$

Proof

First we show that (i) $\Rightarrow$ (ii)

Let $W(0,t_1)=\int_0^{t_1} \Upsilon(t_1,t)\Upsilon^T(t_1,t)dt.$. Define the operator $K: L_\infty([0,t_1],E^m)\to E^n$ by

$$K(u)=\int_0^{t_1} \Upsilon(t_1,t)u(t)dt$$

K is a continuous linear operator from a Hilbert space to another. Thus $R(K)\subseteq E$ is a linear subspace and its orthogonal complement satisfies the relation

$$\{R(K)\}^\perp = N(K^*)$$

where K^* is the adjoint of K .

$$K^*: E^n \to IU\subseteq L_\infty .$$

By the nonsingularity of the controllability grammian, the symmetric operator $KK^T=W(0,t_1)$ is positive definite and hence $\{R(K)\}^\perp=\{0\}$, i.e. $N(K^*)=\{0\}$

For any $c\in E^n, u\in L_\infty$ the inner product $\langle c,Ku\rangle=\langle K^*c,u\rangle$

$$\langle c,Ku\rangle=\left\langle c,\int_0^{t_1} Y(t_1,t)u(t)dt\right\rangle=\int_0^{t_1} c^T\left[\Upsilon(t_1,t)\right]u(t)dt$$

Thus, K^* is given by $c\to c^T[Y(t_1,t)]; t\in[0,t_1]$.

$N(K^*)$ is therefore the set of all $c\in E^n$ such that $c^T[Y(t_1,t)]=0\ a.e. in\,[0,t_1]$. Since $N(K^*)=\{0\}$, all such c are equal to zero i.e. $c=0$. This establishes the properness of system (1.8).

Next we show that (ii) $\Rightarrow$ (iii)

We now show that if system (1.8) is proper then it is relatively controllable on each interval $[0,t_1]$.

Let $c\in E^n$, if system (1.8) is proper then $c^T[Y(t_1,t)]=0\ a.e. t\in[0,t_1]$ for each t_1 implies $c=0$.

Thus

$$\int_0^{t_1} c^T\left[\Upsilon(t_1,t)\right]u(t)dt=0 \text{ for } u\in L_\infty .$$

It follows that the only vector orthogonal to the set

$$IR(t_1,0)=\left\{\int_0^{t_1} \Upsilon(t_1,t)u(t)dt : u\in L_\infty\right\}$$

is the zero vector. Hence $\{IR(t_1,0)\}^{\perp}=\{0\}$, i.e. $IR(t_1,0)=E^n$. This establishes relative controllability on $[0,t_1]$ of system (1.8).

Finally, we show that (iii) $\Rightarrow$ (i)

We now show that if the system (1. 8) is relatively controllable, then $W(0,t_1)$ is nonsingular. Let us assume for contradiction that $W(0,t_1)$ is singular. Then there exists an n vector

$v\neq 0$ such that

$$vWv^T=0.$$

Then

$$\int_0^{t_1}\left\| v\left[\Upsilon(t_1,t)\right]\right\|^2 dt=0.$$

This implies that

$$\left\| v\left[\Upsilon(t_1,t)\right]\right\|^2=0$$

hence

$$v[Y(t_1,t)]=0 \ a.e. \text{ for } t\in[0,t_1].$$

This implies that $v\neq 0$, which contradicts the assumption of properness of the system. This completes the proof.

We also have the following:

***Theorem* 2.2**

System (1.8) *is relatively controllable if and only if* $0\in IntIR(t_1,0)$ *for each* $t_1>0$.

Proof

It is known [4, Corollary 9.2] that $IR(t_1,0)$ is a closed and convex subset of E^n. Therefore a point y_1 on the boundary of $IR(t_1,0)$ implies there is a support plane π of $IR(t_1,0)$ through y_1. This means that $c^T(y-y_1)\le 0$ for each $y\in IR(t_1,0)$ where $c\neq 0$ is an outward normal to π. If u_1 is the control corresponding to y_1 we have

$$c^T\int_0^{t_1}\left[\Upsilon(t_1,t)\right]u(t)dt\le c^T\int_0^{t_1}\left[\Upsilon(t_1,t)\right]u_1(t)dt$$

for each $u\in IU$ where $IU=L_\infty([t_0,t_1],C^m)$. Since IU is a unit sphere, this last inequality holds for $u\in IU$ if and only if

$$c^T\int_0^{t_1}\left[\Upsilon(t_1,t)\right]u(t)dt \ \le \int_0^{t_1} c^T\left[\Upsilon(t_1,t)\right]u(t)dt \ =\int_0^{t_1}\left|c^T\Upsilon(t_1,t)\right|dt$$

and

$$u_1(t)=\operatorname{sgn} c^TY(t_1,t)$$

as y_1 is on the boundary. Since we always have $0\in IR(t_1,0)$, if 0 were not in the interior of $IR(t_1,0)$, then it is on the boundary. Hence from the preceding argument, this implies that

$$0=\int_0^{t_1}\left|c^T\Upsilon(t_1,t)\right|dt$$

so that

$$c^TY(t_1,t)=0 \ a.e.\ t\in[0,t_1].$$

This by definition of properness of system (1.8) implies that the system is not proper since $c\neq 0$, hence if $0\in IntIR(t_1,0)$

$$c^TY(t_1,t)=0 \ a.e.\ t\in[0,t_1]$$

would imply $c=0$ proving properness and by Theorem 2.1, it is concluded that system (1.8) is relatively controllable for each interval $[0,t_1]$.

***Proposition* 2.1**

If system (1.8) is relatively controllable on $[0,t_1], t_1 > 0$, then D, the domain of null controllability of (1.8) contains zero in its interior**.**

Proof

Observe that $0 \in IntD$, since $x(t) = 0$ is a solution of system (1.8) with $u = 0$. Assume that system (1.8) is relatively controllable on $[0,t_1], t_1 > 0$, then by Theorem 2.2, $0 \in IntIR(t_1,0)$ for each $t_1 > 0$. Suppose 0 is not the interior of D, then there exists a sequence $\{x_m\}_1^\infty \subseteq C$ such that $x_m \to 0$ as $m \to \infty$ and no x_m is in D (so $x_m \neq 0$).

From the variation of constant formula, we have

$$0 \neq x(t_1,0,x_m,u) = X(t_1,0)x_m(0) + \sum_{i=0}^{p}\int_{-h_i}^{0} X(t_1,t+h_i)A_i(t+h_i)x_m(t)dt$$
$$+\sum_{i=0}^{p}\int_{-h_i}^{0} X(t_1,t+h_i)B_i(t+h_i)u_0(t)dt + \sum_{i=0}^{p}\int_{0}^{t_1-h_i} X(t_1,t+h_i)B_i(t+h_i)u(t)dt$$

for any $t_1 > 0$ and any $u \in IU$.

Define $r_m = x(t_1,0,x_m,0) = X(t_1,0)x_m(0) + \sum_{i=0}^{p}\int_{-h_i}^{0} X(t_1,t+h_i)A_i(t+h_i)x_m(t)dt$. Hence $r_m \notin IR(t_1,0)$ for any $t_1 \geq 0$. Therefore the sequence $\{r_m\}_1^\infty \subseteq E^n$, $r_m \in IR(t_1,0), r_m \neq 0$ is such that $r_m \to 0$ as $m \to \infty$. There $0 \notin IntIR(t_1,0)$ for any t_1, a contradiction. Hence, $0 \in IntD$.

In the next section we harness the results put together above to establish the main result of this paper.

3.0 Main result

***Theorem* 3.1**

In (1.8) *assume that*

(i) *system* (1.8) *is relatively controllable on* $[0,t_1]$ *for each* $t_1 > 0$;

(ii) *the zero solution of system (*1.8) *with* $u = 0$ *is uniformly asymptotically stable, so that the solution of (*1.8) satisfies

$$\|x_t(\phi)\| \leq M\|\phi\|e^{\infty t}, t \geq 0$$

$\alpha > 0, M > 0$ are constants

then system (1.8) is relatively null controllable with constraints.

Proof

By condition (*i*), system (1.8) is relatively controllable with constraints, so that $0 \in IntIR(t_1,0), t_1 > 0$. Then by Proposition 2.1, the domain D of relative null controllability of system (8) contains zero in its interior. Therefore, there exists a ball P such that $0 \in P \subseteq D$. By condition (ii), every solution of (1.8) with $u = 0$ satisfies $x(t,0,\phi,0) \to 0$ as $t \to \infty$. Hence at some $t_1 < \infty, x(t,0,\phi,0) \in P \subseteq D$ for $t_1 > 0$. Therefore, using t_1 and $x_1 = x(t_1,0,\phi,u)$ as initial data, there exists a $u \in IU$ and some $t_2 > t_1$ such that the solution $x(t,t_1,x_1,u)$ of (1.8) satisfies $x(t_2,t_1,x_1,u) = 0$. Hence system (1.8) is relatively null controllable.

***Theorem* 3.2**

Assume

(i) *system* (1.8) *is relatively null controllable on* $[0,t_1]$ *for each* $t_1 > 0$;

(ii) *the zero solution of* (1.8) *with* $u = 0$ *is uniformly asymptotically stable*; *then system* (1.8) *is null controllable with constraints.*

Proof

Condition (i) and Theorem 3.1 guarantee an open ball $N \subseteq D$ such that every $\phi \in N$ can be steered to zero point of E^n with controls from IU in time $t_1 < \infty$. Condition (ii) ensures that every solution of system (1.8) with $u = 0$ satisfies $x(t,0,\phi,0) \to 0$ as $t \to \infty$. Thus using $u = 0$, there exists a $t_2 < \infty$, such that $x_2 = x(t_2, 0, \phi, 0) \in N$. With x_2 and t_2 as initial data, there exists $t_3 < t_2$ such that for some $u \in IU, x(t_2, t_2, x_2, u) = x_2, x(t_3, t_2, x_2, u) = 0$
Thus the control

$$w = \begin{cases} 0 & in[0, t_2] \\ u & in[t_2, t_3] \end{cases}$$

transfers ϕ to the origin in $t_1 < \infty$. This completes the proof.

4.0 Conclusion

From the sequel, sufficient conditions for the relative null controllability of linear systems with time varying multiple delays in state and control have been developed. These results are given with respect to the stability of the free linear base system and the relative controllability of the linear controlled system. computable criteria for all these are reported. This work extends known results in the literature to linear systems with multiple delays in both the state and control variables.

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