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Biomechanics of the brain; some remarks on Biot's equations of consolidation theory with deformation-dependent permeability

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Abstract

We revisit models of hydrocephalus in the literature. In particular, we examine the class of models based on Biot's theory of consolidation with fixed boundary forcing. Instead of fixed boundaries we take free boundaries. We prove existence and uniqueness of solutions. As in the fixed boundary forcing, we show that in a free boundary, the pressure is higher when the permeability depends on deformation. On the other hand, the total filtration is lower. Unlike the fixed forcing, the effect of the deformation on permeability reduces over time:

***Keywords*: Consolidation theory, variable permeability, free boundary, upper and lower solutions.**

1.0 Introduction.

Models of hydrocephalus, which is characterized by excessive accumulation of fluid in the brain, have appeared in the literature over the past 2 decades ([1] – [6]). The fundamental premise of these models is that a correct description of hydrocephalus must take into account the sponge – like physical constitution of the brain parenchyma. Based on this, the ventricular enlargement is assumed to be the result of the squeezing out of the fluid from the microscopic pores of the brain parenchyma as a consequence of the hydrodynamic loading due to an increase in the ventricular cerebrospinal fluid pressure, whilist, the accumulation of the fluid observed in periventricular edema can be explained as the result of filtration through the ventricular walls [4]. Hence the brain is treated as a porous medium.

In a recent paper Sivaloganathan et al. [4] derived a single non-linear parabolic equation for the unidirectional deformation of the brain tissue.

In this paper, we start with the single equation and instead of fixed boundaries we assume free boundaries and we investigate the properties of the pressure and filtration. Of particular interest is the structure of the spatial and temporal solutions.

2.0 Mathematical Model

The single equation that controls pressure and velocity profiles is

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial}{\partial x}\left(e^{-cv}\frac{\partial v}{\partial x}\right), \quad c \geq 0, \quad t > 0 \text{ (see[4])} \tag{2.1}$$

where $$v = \frac{f(t) - \rho}{\alpha} \quad (2.2)$$

$$\frac{\partial u}{\partial x} = \frac{\rho - f(t)}{\alpha} \quad (2.3)$$

Here

ρ	is pressure
u	is velocity of the cerebrospinal fluid
$f(t)$	is the forcing function
x	is the space variable
t	is time
α	is the Lame coefficient
c	measures the dependence of permeability on deformation.

Sivaloganathan et al. [4] use the following for boundary and initial conditions

$$v(o,t) = 0 \quad (2.4)$$

$$v(L,t) = \frac{f(t)}{\alpha} \quad (2.5)$$

$$v(x,0) = 0 \quad (2.6)$$

(2.4) and (2.5) assume a fixed boundary. In this paper, we assume moving boundaries

$$x_0(t) = \lambda A t^{1/2}, \quad x_f(t) = A t^{1/2}, \quad 0 \le \lambda < 1$$

where x_o is the source of forcing function and x_f denotes the final boundary Both boundaries are functions of time. In this paper our boundary and initial conditions are

$$v(x_o,t) = 0, \; t > 0 \quad (2.7)$$

$$v(x_f,t) = \frac{p_0}{\alpha}, \; t > 0 \quad (2.8)$$

$$v(x,t) = 0, \; t \le 0 \quad (2.9)$$

3.0 Similarity Solution

Let the similarity variable be

$$\eta = \frac{x}{At^{1/2}}, \quad v = f(\eta)$$

So, (2.1) becomes $$-\frac{1}{2}\eta f^1 = \frac{\alpha}{A^2}\frac{d}{d\eta}\left(e^{-cf}\frac{df}{d\eta}\right), \quad \lambda \le \eta \le 1 \quad (3.1)$$

that is, $$\frac{d^2 f}{d\eta^2} + \frac{1}{2}\eta f^1 = \frac{c^{\alpha}}{A^2} e^{-cf}\left(\frac{df}{d\eta}\right)^2 \quad (3.2)$$

$$f(\lambda) = 0, \; t > 0 \quad (3.3)$$

$$f'(\lambda) = \sigma, \; t > 0 \quad (3.4)$$

$$f(1) = \frac{p_0}{\alpha}, \; t > 0 \quad (3.5)$$

We imposed boundary condition (3.4) to ensure a unique solution. When (3.4) is used, σ is guessed so that boundary condition (3.5) is finally satisfied. We continue to change σ until (3.5) is satisfied.

We are interested in a solution f(η) for which

$$\lambda \le \eta \le 1, \quad \left| f(\eta) - \frac{\rho}{\alpha} \right| \le a, \; |f'(\eta) - \sigma| \le a \quad (3.6)$$

That is, both f and its derivative $f'(\eta)$ are bounded.

4.0 Existence and uniqueness of solution

***Theorem* 4.1**

Problem (3.2) which satisfies (3.3), (3.5) and (3.6) has a unique solution

Proof:

Let $x_1 = \eta$

$x_2 = f(\eta)$

$x_3 = f'(\eta)$

Then

$$\left.\begin{aligned} x_1' &= 1 \\ x_2' &= x_3 \\ x_3' &= c\, e^{x2} x_3^2 - \frac{1}{2}\frac{A^2}{\alpha} x_1 x_3 \end{aligned}\right\} \qquad (4.1)$$

Thus,

$$\left.\begin{aligned} x_1' &= f_1(x_1,\ x_2,\ x_3) = 1 \\ x_2' &= f_2\ (x_1,\ x_2,\ x_3) = x_3 \\ x_3' &= f_3(x_1,\ x_2,\ x_3) = c\, e^{cx2} x_3{}^2 - \frac{1}{2}\frac{A^2}{\alpha} x_1 x_3 \end{aligned}\right\} \qquad (4.2)$$

Hence $\dfrac{\partial f_i}{\partial x_j}$, $i = 1,\ 2,\ 3,\ j = 1,\ 2,\ 3$ are continuous and bounded. Hence by Theorem 11.7 [2] problem (3.1) which satisfies (3.2), (3.4) and (3.5) has a unique solution

5.0 Properties of the Solution

The disease free case is given by c=0. In this case.

$$\frac{d^2 f}{d\eta^2} + \frac{A^2}{2\alpha}\frac{\eta\, df}{d\eta} = 0 \qquad (5.1)$$

As shown in the appendix thus a solution of (3.1) is a lower solution for (5.1) since $c \geq 0$ and $\dfrac{c\alpha}{A^2} e^{-cf}\left(\dfrac{df}{d\eta}\right)^2 \geq 0$

Now $f(\eta)$ is a lower solution implies that $v(x,t)$ is a lower solution. Thus $p(x, t)$ is higher for $c > 0$ than for $c = 0$ by equation (2.2). Also u_x is lower for $c > 0$ than for $c = 0$ by equation (2.3).

We have therefore obtained the same results of Sivaloganathan et al. [7] by method of upper and lower solutions. These important results are the following: Variable permeability ($c > 0$) increases the pressure of the cerebrospinal fluid and reduces its velocity.

6.0 Numerical computation

We display in Figures 1-3 the graph of $f(\eta)$ against η for various values of λ. In each case, a solution exists.

7.0 Discussion of the result

Now

$$f(\eta) = \frac{\rho_0 - \rho}{\alpha}, \quad u_x = \frac{\rho_0 - \rho}{\alpha}.$$

There is no steady state. When t is large, $f(\eta) \equiv 0$ so the pressure is constant and cerebrospinal fluid moves at constant velocity.

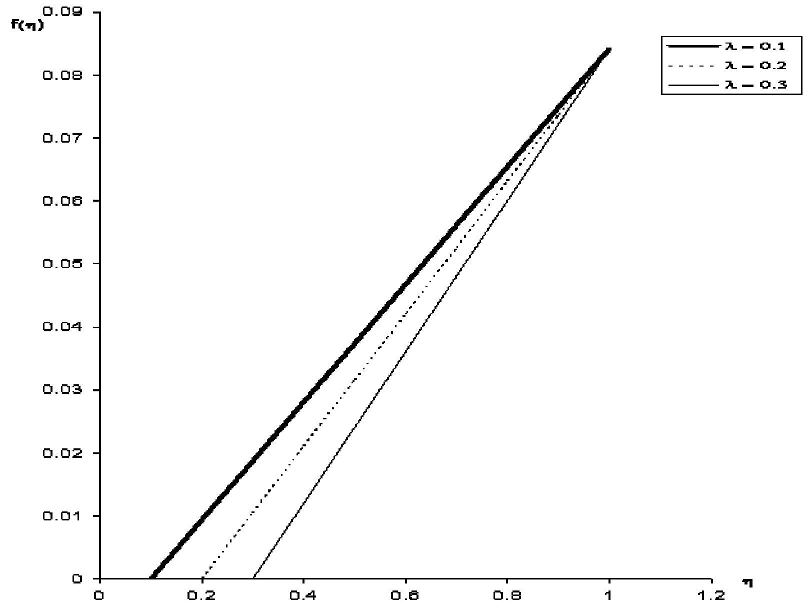

Figure 1: The graph of f(η) against η for fixed values c = 0, A= 1, Po= 2000, α = 23750 and for various values of λ

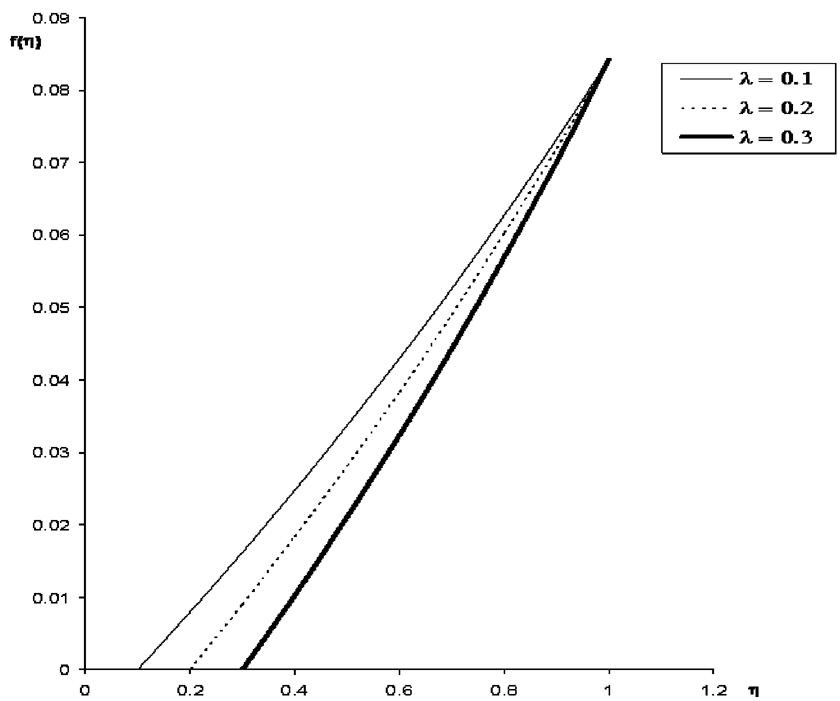

Figure 2: The graph of f(η) against η for fixed values c = 4.3, A= 1 Po= 2000, α = 23750 and for various values of λ

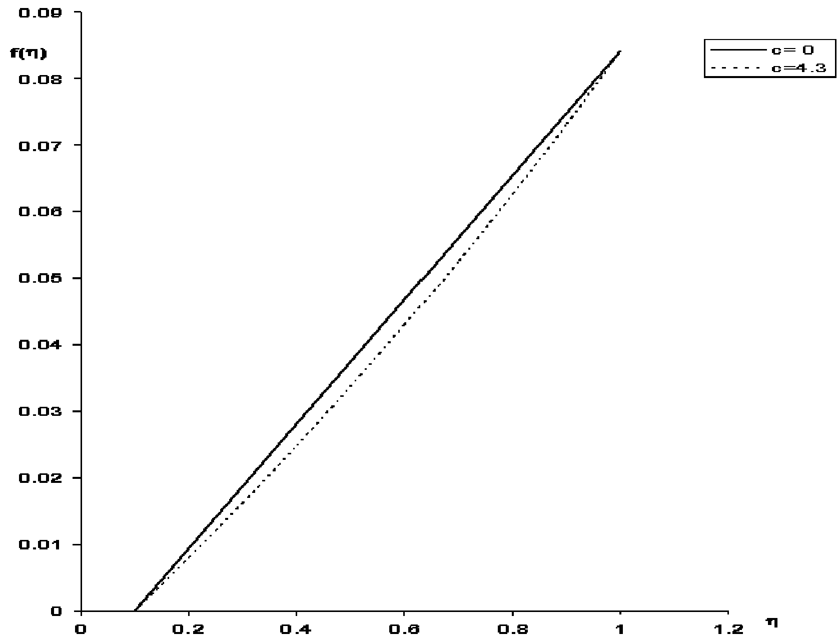

Figure 3: The graph shows f(η) against η for various values of c at λ = 0.1

Appendix

Definition [1]

A smooth function v_0 is said to be an upper solution of the problem (5.1) which satisfies (3.2) and (3.4) if v_0 satisfies

$$\frac{d^2v_0}{d\eta^2} + \frac{A^2\eta}{2\alpha}\frac{dv_0}{d\eta} \leq 0$$

$v_0(\lambda) \geq 0$ $v_0(1) \geq 0$. This implies that $v_0(\eta) \geq f(\eta),\ \lambda \leq \eta \leq 1$

A smooth function u_0 is said to be a lower solution of (5.1) if the inequalities are reversed. In this case

$$u_0(\eta) \leq f(\eta),$$

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