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On a Subclass of analytic functions

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Abstract

Abstract. In this work we establish some conditions for univalence and our results include starlikeness, convexity and close-to-convexity

Keywords*:* Analytic, Univalent, Starlikeness, Convexity, Close-to-convexity Salagean derivative.

1.0 Introduction

Let C be the complex plane. Denote by A the class of normalized functions

$$f(z) = z + a_2 z^2 + ... = z + \sum_{k=m+1}^{\infty} a_k z^k, \qquad m \in N = \{1,2,...\} \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let $\alpha > 0$ be real. Using binomial expansion, we can write

$$f(z)^\alpha = z^\alpha + \sum_{k=m+1}^{\infty} a_k(\alpha) z^{\alpha+k-1}. \tag{1.2}$$

In [4], Opoola introduced and studied the class $T_n^\alpha(\beta)$ consisting of functions $f \in A$ satisfying

$$\operatorname{Re}\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta, \quad \alpha > 0,\ 0 \le \beta < 1, \quad z \in E \tag{1.3}$$

where $D^n (n \in N_0 = \{0,1,2,...\})$ is the Salagean derivative operator define as

$$D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))' \tag{1.4}$$

with $D^0 f(z) = f(z)$.

Note here that the geometric condition (1.3) slightly modifies the one given originally in [4] see [2].

The class $T_n^\alpha(\beta)$ is a very large family of analytic and univalent functions, which has as special cases, many other classes of functions which have attracted the attention of many authors. For instance, several results concerning the cases

(i) $T_0^1(0) \equiv S_0$

(ii) $T_0^1(\beta) \equiv S_0(\beta)$

(iii) $T_1^1(0) \equiv R$

(iv) $T_1^1(\beta) \equiv R(\beta)$

(v) $T_1^\alpha(0) \equiv B_1(\alpha)$

(vi) $T_n^\alpha(0) \equiv B_n(\alpha)$

can be found in the literatures [1,2, 3,4,6,7].

The main object of this paper is to derive certain conditions for univalency of analytic -functions in the unit disk. Our results contain condition for starlikeness, convexity and close- to-convexity of analytic functions in the unit disk.

In order to give our results we have to recall here the following lemma

Lemma **A**. See [5]

Let the (non-constant) function $w(z)$ *be analytic in* E *with* $w(0)=0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z|=r<1$ *at a point* $z_0\in E$, *then*

$z_0 w'(z_0)=cw(z)$ *where* c *is real number and* $c\ge 1$.

2.0 Main Results

***Theorem* 2.1**

Let the function $f\in A$ satisfies the inequality

$$\operatorname{Re}\alpha\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}>\alpha+\frac{1-\beta}{2(1+\beta)},\qquad (\alpha>0,\ 0\le\beta<1,\ z\in E)\tag{2.1}$$

then (i) $\operatorname{Re}\frac{D^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}}>\frac{1+\beta}{2}$, *(ii)* $\left|\frac{\beta\alpha^{n}z^{\alpha}-D^{n}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}-\alpha^{n}z^{\alpha}}\right|<1$ *and* $f\in T_n^{\alpha}(\beta)$

Proof

We begin by defining $w(z)$ by

$$\frac{D^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}}=\frac{\beta+w(z)}{1+w(z)},\quad (w(z)\ne -1, z\in E,\ \alpha>0,\ 0\le\beta<1,\ n=0,1,2,...)\tag{2.2}$$

Then clearly $w(z)$ is analytic in E with $w(0)=0$. We also find from (2.2) that

$$\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}=\alpha+\frac{zw'(z)}{\beta+w(z)}-\frac{zw'(z)}{1+w(z)}\tag{2.3}$$

Since the R.H.S of (2.3) is independent of n we can write (2.3) as

$$\alpha\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}=\alpha+\frac{zw'(z)}{\beta+w(z)}-\frac{zw'(z)}{1+w(z)}\tag{2.4}$$

see [2].

Suppose now that there exists point $z_0\in E$ such that

$$|w(z_0)|=1 \ and\ |w(z)|<1,\ when\ |z|<|z_0|\tag{2.5}$$

Then by applying Lemma A, we have

$$z_0 w'(z_0)=cw(z_0),\quad (c\ge 1,\ w(z_0)=e^{i\theta},\ \theta\in R)\tag{2.6}$$

Thus, we find from (2.4) and (2.6) that

$$\operatorname{Re}\left\{\alpha\frac{D^{n+1}f(z_0)^{\alpha}}{D^{n}f(z_0)^{\alpha}}\right\}=\alpha+\operatorname{Re}\left(\frac{ce^{i\theta}}{\beta+e^{i\theta}}\right)-\operatorname{Re}\left(\frac{ce^{i\theta}}{1+e^{i\theta}}\right)=\alpha+\frac{c(\beta\cos\theta+1)}{\beta^2+2\beta\cos\theta+1}-\frac{c}{2}\le\alpha+\frac{(1-\beta}{2(1+\beta)}.\tag{2.7}$$

which obviously contradict our hypothesis (2.1). It follows that $|w(z)|<1,\quad (z\in E)$.

That is, that $\operatorname{Re}\frac{D^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}}>\frac{1+\beta}{2}$. This evidently complete the proof.

Corollary A. *Let* $f\in A$ *satisfies the inequality*

$$\operatorname{Re}\frac{D^{1}f(z)}{D^{0}f(z)}>\frac{3-\beta}{2(1+\beta)}\tag{2.8}$$

Then (i) $\operatorname{Re}\frac{D^{0}f(z)}{z}>\frac{1+\beta}{2}$

(ii) $\left|\frac{\beta z-f(z)}{f(z)-z}\right|<1$

and $f\in T_0^{1}(\beta)$.

Proof

The proof is the immediate consequence of putting $n=0$ *and* $\alpha=1$ in Theorem 2.1.

***Corollary* B**

Let the function $f\in A$ *satisfiy the inequality*

$$\operatorname{Re}\frac{D^2 f(z)}{D^1 f(z)} > \frac{3-\beta}{2(1+\beta)}, \qquad (0 \le \beta < 1,\ z \in E) \tag{2.9}$$

then (i) $\operatorname{Re}\frac{D^1 f(z)}{z} > \frac{1+\beta}{2} \Rightarrow \operatorname{Re} f'(z) > \frac{1+\beta}{2}$

(ii) $\left|\frac{\beta - f'(z)}{f'(z)-1}\right| < 1$

and $f \in T_1^1(\beta)$.

Proof

The proof is the immediate consequence of putting $n = 1$ *and* $\alpha = 1$ in Theorem 2.1

***Theorem* 2.2**

Let the function $f \in A$ *satisfy the inequality*

$$\operatorname{Re}\alpha\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} > \frac{2\alpha-3}{2} - \frac{1-3\beta}{2\beta^2}. \tag{2.10}$$

Then (i) $\operatorname{Re}\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta$ (2.11)

(ii) $\left|\frac{\alpha^n z^\alpha - D^n f(z)^\alpha}{D^n f(z)^\alpha - (1-2\beta)\alpha^n z^\alpha}\right| < 1$

and $f \in T_n^\alpha(\beta)$.

Proof

We begin by defining a function $w(z)$ by

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1-(1-2\beta)w(z)}{1+w(z)}, \quad w(z) \ne -1, z \in E, \alpha > 0,\ 0 < \beta < 1. \tag{2.12}$$

Then, clearly, $w(z)$ is analytic in E with $w(0) = 0$. We also find from (2.12) that

$$\alpha\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha - \frac{(1-2\beta)zw'(z)}{1-(1-2\beta)w(z)} - \frac{zw'(z)}{1+w(z)}. \tag{2.13}$$

Suppose now that there exists a point $z_0 \in E$ such that

Then, by applying Lemma A, thus we have from (2.12) and (2.6) that

$$\operatorname{Re}\left\{\alpha\frac{D^{n+1} f(z_0)^\alpha}{D^n f(z_0)^\alpha}\right\} = \alpha - \operatorname{Re}\left(\frac{c(1-2\beta)e^{i\theta}}{1-(1-2\beta)e^{i\theta}}\right) - \operatorname{Re}\left(\frac{ce^{i\theta}}{1+e^{i\theta}}\right) \tag{2.14}$$

$$= \alpha - \frac{c(1-2\beta)[1-2\beta+\cos\theta]}{1-2(1-2\beta)\cos\theta+(1-2\beta)^2} - \frac{c}{2} \le \frac{2\alpha-3}{2} - \frac{(1-3\beta)}{2\beta^2}, \qquad (z_0 \in E,\ \alpha > 0,\ 0 < \beta < 1)$$

which obviously contradicts our hypothesis (2.10). It follows that $|w(z)| < 1$, $(z \in E)$. That is, that

$$\left|\frac{\alpha^n z^\alpha - D^n f(z)^\alpha}{D^n f(z)^\alpha - (1-2\beta)\alpha^n z^\alpha}\right| < 1, \quad (z \in E,\ \alpha > 0,\ 0 < \beta < 1).$$

This evidently completes the proof of Theorem 2.2

Corollary *C*

Let the function $f \in A$ *satisfy the inequality*

$$\operatorname{Re}\alpha\frac{D^1 f(z)^\alpha}{D^0 f(z)^\alpha} > \frac{2\alpha-3}{2} - \frac{(1-3\beta)}{2\beta^2} \tag{2.15}$$

then

(i) $$\text{Re}\frac{D^0 f(z)^\alpha}{z^\alpha} > \beta \tag{2.16}$$

(ii) $$\left|\frac{z^\alpha - f(z)^\alpha}{f(z)^\alpha - (1-2\beta)z^\alpha}\right| < 1$$

and $f \in T_0^\alpha(\beta)$.

Proof

Corollary C is the immediate consequence of putting $n = 0$ in Theorem 2.2

***Corollary* D**

Let the function $f \in A$ satisfy the inequality

$$\text{Re}\frac{D^1 f(z)}{D^0 f(z)} > \frac{3-\beta}{2} - \frac{1}{2\beta^2} \Rightarrow \text{Re}\frac{zf'(z)}{f(z)} > \frac{3-\beta}{2} - \frac{1}{2\beta^2} \tag{2.17}$$

then (i) $$\text{Re}\frac{D^0 f(z)}{z} > \beta \Rightarrow \text{Re}\frac{f(z)}{z} > \beta \tag{2.18}$$

(ii) $$\left|\frac{z - f(z)}{f(z) - (1-2\beta)z}\right| < 1$$

and $f \in T_0^1(\beta)$.

Proof

The proof of Corollary D is the immediate consequence of putting $n = 0$ *and* $\alpha = 1$ in Theorem 2.2

***Corollary* E**

Let the function $f \in A$ satisfying the inequality

$$\text{Re}\frac{D^2 f(z)}{D^1 f(z)} > \frac{3-\beta}{2} - \frac{1}{2\beta^2} \Rightarrow \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3-\beta}{2} - \frac{1}{2\beta^2} \tag{2.19}$$

then $$\text{Re}\, f'(z) > \beta \tag{2.20}$$

and $f \in T_1^1(\beta)$

Proof

Putting $n = 1$ *and* $\alpha = 1$ in Theorem 2.2, the result follows.

3.0 Conclusion

In this work we are able to generalized the subclasses of analytic functions earlier mentioned on page one.

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