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Periodic solutions of periodic differential equations

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Abstract

In this paper we extend the work of Bello [4] where he considered the periodic solutions of certain dynamical systems inside a cylindrical phase space with differential equations of the form

$$y^{(n-1)} + \alpha_1 y^{(n-1)} + \ldots + \alpha_{n-1} y^{(1)} + f\left(y^1 \ldots, y^{(n-1)}, y\right) = 0 \quad \left(' = \tfrac{d}{dt}\right) \qquad \textbf{(+)}$$

with the property that there is a $w > 0$ and a natural number K such that

$$y(t+w) = y(t) + k, \ \forall t \qquad \textbf{(**)}$$

with necessary and sufficient condition that the fundamental matrix $\phi(\lambda)$ of the characteristic equation

$$\phi(\lambda) \ \ \lambda^{n-1} + \alpha_1 \lambda^{n-2} + \ldots + \lambda_{n-1} = 0 \qquad \textbf{(***)}$$

of (*) have negative real parts (See [1], [7]), $\phi(\lambda)$ is stable asymptotically. The extension considered the periodic solutions of the differential equations of the type

$$y' + \lambda(y')y'' + b(y)y' + f(y) = 0 \qquad \textbf{(****)}$$

with the property (**). The periodic solutions and the asymptotic behaviour of the solutions were investigated and analysed. Some theorems were proved and examples given to illustrate certain properties of the solutions.

1.0 Introduction

Many authors (See [3], [4] and [5]) considered the periodic solutions of differential equations of the type

$$y''' + a(y')y'' + by' + f(y) = 0 \qquad (1.1)$$

where $a, b > 0$ and *f* is a class function C' and of period 1 with the property

$$y(t+w) = y(t) + 1, \ \forall t \qquad (1.2)$$

for every $w > 0$.

In their methods of approach, they proved some theorems to illustrate certain properties of the solution which included Schauder's Fixed Theorem and an alternative method (See [5]) as used in Laser.
The existence of periodic solutions which are even or odd functions was also discussed (See [5]). This work also served as the generalisation of the results of Bello [4], Chang [5] and Nazarov [9]. Some Lemmas were stated and theorems proved.

2.0 Preliminaries

We considered the existence of periodic solutions of the differential equation of the type

$$y' + a(y') + b(y)y' + f(y) = 0 \qquad (2.1)$$

(See [5]) and established the property (1.2) with additional provision that *k* is now substituted for 1.

The analytic solution (2.1) was given and a numerical example used as illustration.

3.0 Periodic Solutions of Differential Equations in the Cylindrical Space

It has been proved by Bello [4] that if the function f defined on the real line R is negative, continuous and periodic and if the polynomial

$$\phi(\lambda) = \lambda^{n-1} + \alpha_1 \lambda^{n-2} + \ldots + \alpha^{n-1} \qquad (3.1)$$

of the differential equation

$$\frac{d^n y}{dt^n} + \alpha_i \frac{d^{n-1} y}{dt^{n-1}} + \alpha_{n-1}\frac{dy}{dt} + f\left(y' \ldots y^{n-1}, y\right) \qquad (3.2)$$

has all the roots with negative real parts, the equation (3.2) admits at least one solution $y = y(t)$ having the property that there is a $t > 0$, such that

$$y(t+w) = y(t) + k, \text{ for } t \in R. \qquad (3.3)$$

It will be shown in this paper that this result holds true under the assumption that (3.1) has roots with negative real parts.

The existence of a solution with a property (3.3) will be proved for the equation

$$\frac{d^3 y}{dt^3} + a\left(\frac{dy}{dt}\right)\frac{d^2 y}{dt^2} + b(y)\frac{dy}{dt} + f(y) = 0 \qquad (3.4)$$

Let $x(t)$ be the $(n-1)$–vector with components $\left(y'(t), y''(t), \ldots, y^{(n-1)}(t)\right)$ as a solution of the equation (3.2). It is clear that this is equivalent to the property which is such that there is $w > 0$ with

$$x(t+w) = x(t), \quad y(t+w) = y(t) + 1 \quad \forall t \qquad (3.5)$$

***Theorem* 3.1**

Let R_c^n denote the set obtained from the n-dimensional Euclidean space R by the identification of pairs $(x_1, \ldots, x_{n-1}, u)$, $(x_1, \ldots, x_{n-1}, v)$ with integer $u - v$, the "cylindrical space" (See [14]) then implies that the map $t \to (x(t), y(t))$ of R into R_c^n is periodic with period w.

Write (3.4) as a system (See [3])

$$\left\{\begin{array}{l} x' = P(x, y) \\ y' = x_1 \end{array}\right\} \qquad (3.6)$$

in which $(x_1, \ldots, x_{n-1})$ and $P : R^{n-1} \times R \to R^n$ is continuous and periodic in y with period 1.

It will be shown under the assumption of the two theorems later in this paper that the system (3.4) admits a solution satisfying the property (3.5)

The following Lemma will be used in proving the required theorems.

***Lemma* 3.1**

Assume that for an arbitrary $(x_o, y_o) \in R$, the initial value problem (3.6) $x(0) = x_o$, $y(0) = y_o$ *has a unique solution* $x = x(t, x_o, y_o)$, $y = y(t, x_o, y_o)$ *which exists for all* $t > 0$.

Let the map $R^{n-1} \to R$ be continuous and denote by S_o, S_1 the sets

$$S_o = \left\{(x, y) : y = h(x), \; x \in R^{n-1}\right\}, \qquad (3.7)$$

$$S_1 = \left\{(x, y) : y = h(x) + 1, \; x \in R^{n-1}\right\}, \qquad (3.8)$$

For any $x_0 \in R^{n-1}$, there is a unique number $t(x_0)$ such that

$$\left(x(t(x_0); x_0, h(x_0)), \; y(t(x_0); x_0, h(x_0)) \right) \in S_1 \qquad (3.9)$$

Then, the system (3.6) has a solution satisfying (3.5) if and only if map $U : R \to R$ defined by

$$U(x_0) = x(t(x_0); x_0, h(x_0)) \qquad (3.10)$$

has a fixed point.

As usual, the following rotation will be used: If A is an n x m-matrix, then $\|A\|$, A^T denote respectively the norm and the matrix transpose of *A*. *FrB*, (C,B) denotes the boundary (closure) of the set B.

For a real-valued function $W(x,y)$, $\left(x\in R^{n-1}, y\in R\right)$ denotes its derivatives with respect to the solutions of (3.6) i.e

$$W'(x,y)=\frac{dw}{dt}(x(t),y(t))$$

where $x(t)$, $y(t)$ is a solution of the system (3.6).

***Theorem* 3.2**

Let a map $f: R\to(-\infty,0)$ *be continuous and let*

$$f(y+1)=f(y) \text{ for } y\in R \tag{3.11}$$

If the polynomial (3.1) *has no pure imaginary roots then the equation* (3.2) *admits, at least one solution* $y=y(t)$ *having the property that there is a* $w>0$ *such that the property* (3.3) *holds*

Proof:

Let the system (1.2) be in the form

$$\begin{cases} x'=Ax+bf(y) \\ y'= \quad x_1 \end{cases} \tag{3.12}$$

where $A=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \dots & 0 & 1 \\ 0 & & & \dots & 0 & 1 \\ -\alpha_{n-1} & 0 & 0 & \dots & 0 & -\alpha_1 \end{bmatrix}$, $b=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$ and $x=\begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$

Let S_0 and S_1 be defined as in the Lemma 3.1, with $h(x)=(-\alpha_{n-1})^{-1}d^Tx$, that is

$$y=(-\alpha_{n-1})^{-1}d^Tx \tag{3.13}$$

$$y=h(x)+1 \tag{3.14}$$

and with

$$d^T=[\alpha_{n-2},\dots,\alpha_1,1] \tag{3.15}$$

we obtain,

$$W(x,y)=\alpha_{n-1}y+d^Tx \tag{3.16}$$

Since f is continuous, f is bounded [9] and this implies that the solutions of the system (3.12) exists for all t because of (3.1). Since $h(x)=(-\alpha_{n-1})^{-1}d^Tx$ then by (3.13), (3.14) becomes

$$W(x,y)=\alpha_{n-1}y+d^Tx=\alpha_{n-1}\left(\frac{-d^T+\alpha_{n-1}}{\alpha_{n-1}}\right)+d^Tx=-d^Tx+\alpha_{n-1}+d^Tx=\alpha_{n-1}$$

By (3.17) and the equation (3.18), the equation

$$W\left(x(t;x_0,h(x_0)),\ y(t;x_0,h(x_0))\right)=\alpha_{n-1} \tag{3.20}$$

has a unique solution $t(x_0)$ for every $x_0\in R$, thus (3.9) holds.

Assume additionally that the system (3.12) has the property of uniqueness. Thus the Lemma 3.1 is applicable and the proof reduces to showing that the mapping U has a fixed point. (See [5], [6], [12]).

Let the map U be defined by the formula

$$U(x_0)=P(x_0)x_0+b(x_0) \tag{3.21}$$

where $P(x_0)=(t(x_0))$ and

$$b(x_0)=\int_0^{t(x_0)}x(t(x_0)-s)\ bf(y(s,x_0,h(x_0)))ds \text{ (See [9] and [12])} \tag{3.22}$$

Here $x(t)$ denotes the fundamental matrix of $x' = Ax$ which implies that $U(x_0)$ is the solution of the system (3.12). Since A has no eigenvalues on the imaginary axis, $P(x_0)-1$ is non-singular. We now show that $\|b(x)\| = 0$ as $\|x\| \to \infty$. From the integral $b(x_0) = \int_0^{t(x_0)} x(t(x_0)-s)\ bf(y(s, x_0, h(x_0)))ds$ we have

$$\|b(x_0)\| = \left\| \int_0^{t(x_0)} x(t(x_0)-s)\ bf(y(s, x_0, h(x_0)))ds \right\| \leq \int_0^{t(x_0)} \|x(t(x_0)-s)\|\ b\int_0^{t(x_0)} \|f(y(s, x_0, h(x_0)))\| ds \leq \|x(t)\|\ bf(y)$$

since $b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$,

Hence,

$$\|b(x)\| \leq \|x(t)\| \|f(y)\| \qquad (3.24)$$

because of the boundedness of $f(y)$.

But $U(x) \to 0$ as $x \to \infty$ and so we obtained

$$\lim_{\|x\| \to \infty} \left(\|b(x)\| \bullet \|x\|^{-1} \right) = 0, \ \|f(y)\| \leq 0 \qquad (3.25)$$

Since $t(x)$ is a unique solution of (3.21) then $t(x)$ is also bounded, as $f(y)$ and with boundedness of $t(x)$ and $f(y)$, and by the finite-dimensional theorem (see [8] and [12]), the map $U : R^{n-1} \to R^{n-1}$ is one to one, hence U has a fixed point and we complete the proof of Theorem 3.2.

In the general case, we observe that the boundedness of $\|P(x)-1\|$ and $\|b(x)\|$ led to the fact thus,

$$\{x \in R^{n-1} : U(x) = x\} \qquad \{x \in R : \|x\| < k\} \qquad (3.26)$$

where a constant K>0 depends on the matrix A and estimates f. This permits us to approximate with the uniqueness property such that the corresponding sets of fixed points are in the ball.

$$\{x : \|x\| < k\} \text{ (See [4])} \qquad (3.27)$$

By the standard limiting argument Theorem [7], we can conclude that the Theorem 3.1 holds without assumption that the solution is unique.

4.0 Basic Theorem

The following theorem will also be proved in this section, the assumption of which we would use to show that the system (3.6) admits a solution satisfying the property (3.8)

***Theorem* 4.1**

Let f: $R \to (-\infty, 0)$ be continuous and let the property (3.5) *hold. Let the function $a : R \to R$, $b : R \to R$ be continuous and satisfy*

$$|a(y)| \geq 0 \quad \text{for } y \in R, \qquad (4.1)$$

$$b(y+1) = b(y) \ \text{for } \ y \in R \qquad (4.2)$$

Let the solutions of the equation (3.4) *be uniquely determined by initial conditions and exist for all t > 0. Then the equation* (3.4) *has a solution having property* (3.5)

Proof:

Let the system (3.4) be written as

$$\begin{cases} x_1' = x_2 \\ x_2' = -a(x_1)x_2 - b(y)x_1 - f(y) \\ y' = x_1 \end{cases} \qquad (4.3)$$

And let $a(x)$, $b(x)$ be positive, put

$$W(x,y) = x_2 + A(x_1) + B(y) \tag{4.4}$$

and $h(x) = \left(B^{-1} - x_2 - A(x_1)\right)$ (4.5)

Then from (3.18) and (4.4) we obtain $B(h(x)) = -x_2 - A(x_1)$

where $x = (x_1, x_2)$, $A(z) = \int_0^z a(s)ds$, $B(z) = \int_0^z b(s)ds$

By the Theorem 3.1 and the system (4.4) we have from (4.6) and (4.7)

$$A(z) = \int_0^z a(s)ds \quad = \int_0^x x_2 dx = x_2^2 / 2 \tag{4.8}$$

and

$$B(z) = \int_0^z b(s)ds = \int_0^z (-x_2 - A(x_1))dx = \int_0^x (-x_2 - x_2^2/2)dx = -\tfrac{1}{2}x_2^2 - \tfrac{1}{3}x_2^3 \tag{4.9}$$

It follows from (4.4) that,

$$W(x,y) = x_2 - \tfrac{1}{2}x_2^2)dx = -\tfrac{1}{2}x_2^2 - \tfrac{1}{3}x_2^{33} \tag{4.10}$$

Since a is even and is continuous for all x, ([6] & [8] referred) and for the fact that there exists a number $x_0 > 0$ such that $A(x) < 0$ for $0 < x < x_0$ and $A(x) > 0$ and monotonically increasing for $x > x_0$, and furthermore, since $A(x) \to \infty$ as $x \to \infty$, we see that the equation (3.4) possesses an essential unique non-trivial solution by the Theorem 3.1. (See also [3])

Also we see that $b(x)$ is odd, has a continuous derivative for all x and $B(x) \to \infty$ thus the equation (3.4) has a periodic solution.

Now let $x(t : x_0, y_0)$, $y(t, x_0, y_0)$ and U be defined as in the Lemma 2.1, we observed that,

$$S_0 = \{(x,y) : W(x,y) \quad = 0\} \tag{4.11}$$

$$S_1 = \{(x,y) : W(x,y) \quad B(1)\} \tag{4.12}$$

Since $x \in R^{n-1}$ and $y \in R$ and because $y - h(x) + 1$, $B(h(x)) = -x_2 - A(x_1)$.

But $W'(x,y) = 1 - 2x^2_{2/2} - 3x^2_{2/3} = 1 - x_2 - x_2^2$

$$= -f(y) \geq \overline{\alpha} > 0 \tag{4.13}$$

That is, $W' = -f(y) \geq \overline{\alpha} > 0$ by (3.19)

This implies that for any $x_0 \in R$ there is a $t(x_0) > 0$ satisfying (3.9). This shows that U is defined for all x_0.

Furthermore, U is a homomorphism (See [7]), Preserving the orientation of R^2.

Since $f(y+1) = f(y)$ for any $x_0 \in R^2$, then the points $x_{i+1} = u(x_i) \quad u = 0,1,...$ belong to the set

$$\{x \in R^2 : x = x(t; x_0, h(x_0)), \ t \geq 0\} \tag{4.14}$$

Let D(C) be the "half-cylinder" (See [3]) then

$$\{x, y \in R^2 \times R : V(x) < C, \quad W(x,y) \geq 0\} \tag{4.15}$$

where $V(x) = \tfrac{1}{2}\left((x_1)^2 + (x_2)^2\right) - ex_1x_2$ and e is a constant such that $0 < e < 1$.

Replacing e by a smaller number of necessary then from (4.16) we obtain

$$V'(x) = \tfrac{1}{2}\{2x_1(x_1)' + 2(x_2)(x_2)'\} - e\{x_1(x_2)' + x_2(x_1)'\} = x_1(x_1)' + (x_2)(x_2)' - e\{x_1(x_2)' + x_2(x_1)'\}$$

$$= x_1x_2 + x - a(x_1)x_2 - b(y)x_1 - f(y) - ex_2[-a(x_1)x_2 - b(y)x_1 - f(y)] - ex_2x_2$$

$$= x_1x_2 + x - ax_1(x_2)^2 - b(y)x_1x_2 - x_2 f(y) - ea(x_1)x_2 + eb(y)(x_1)^2 + ex_1 f(y) - e(x_2)^2$$

From there we obtained

$$V'(x) = x_1x_2 + (x_2 - ex_1)(-a(x_1)x_2) - b(y)x_1 - e(x_2)^2 \tag{4.17}$$

Then by (4.15) and (4.17) we got

$$V'\left(\pm\sqrt{2C},\ 0\right) > 0, \ V'\left(0,\ \pm\sqrt{2C}\right) < 0$$

for sufficiently large C.

By inequality (4.18) the set

$$E = \{(x, y) \in FrD(C) : (x(t; x, y), y(t;, x, y)) \in CLD(C)\} \quad (4.19)$$

is nonempty for all small $t > 0$ and has at least two components (See [3] and [5]). If D_i is any component of E, then the set

$$K_i = \left\{ \begin{matrix} (x, y) \in D(C) \cap S_0 : (s; x_0, y_0), \\ y(s; x_0, y_0) \in D_1 \end{matrix} \right\} \quad (4.20)$$

for some s > 0 and{x(t;x,y), y(t;x,y)ϵ D(C) for t ϵ [0,s] } as can be seen is open in S (See [6]).

5.0 Example

Consider the differential equations

$$y'' + p(t)y = 0 \quad (5.1)$$

where $p(t)$ is a real-valued continuous function of period w,

Let $p(t) = \delta^2 + \varepsilon \cos 2\phi$ and write (5.1) in the form

$$y'' + (\delta^2 + \varepsilon \cos 2\phi)y = 0 \quad (5.2)$$

in which the periodic coefficient is a simple harmonic function of the independent variable *t*, δ and ε are real constants.

We will now indicate some of the properties, particularly the stability of the solution of 5.1. We know from Theorem 3.1 that all the solutions of 5.1 are stable since $\phi(t) \leq M$; therefore we shall investigate the conditions on $p(t)$ under which all the solutions of (5.1) are bounded as $t \to \infty$.
To do this, we construct two linearly independent solutions $y_1(t)$ and $y_2(t)$ of (5.6) with the initial condition $y_1(0) = 1$, $y_1''(0) = 0$ and $y_2(0) = 0$, $y_2''(0) = 1$ respectively. (See [6])
In particular, for $\varepsilon = 0$, the solutions of (5.2) are clearly simple harmonic functions and hence bounded as $t \to \infty$.

Since p(t) is a continuous periodic function of period w, $y_1(t+w)$ and $y_2(t+w)$ are also functions of (5.1). Hence any f these solutions can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$. Therefore

$$\left. \begin{matrix} y_1(t+w) = y_1(w) y_1'(w) y_2(t) \\ y_2(t+w) = y_2(w) y_2'(w) y_2(t) \end{matrix} \right\} \quad (5.3)$$

Let $W(t)$ be the Wronskian of $y_1(t)$ and $y_2(t)$. In view of the initial conditions

$$W(t) = 1 \text{ and } W(w) = 1 \quad (5.4)$$

We now found those solutions of (5.1) that have the property

$$y(t+w) = \sigma y(t) \quad (5.5)$$

where σ is a constant.
Such solutions are known as normal solutions, (See [3]). Any normal solution of y(*t*) of (5.1) if it exists, can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \quad (5.6)$$

Where (c_1, c_2) (not both zero) are some suitable constants. From relations (5.3), (5.5) and (5.6), it follows that

$$\left. \begin{matrix} c_1(y_1(w) - \sigma) + c_2(w) = 0 \\ c_1 y_1'(w) + c_2(y_2'(w) - w) \end{matrix} \right\} \quad (5.7)$$

This is a linear homogenous algebraic system and has a non-trivial solution (c_1, c_2) if and only if

$$\begin{vmatrix} y_1(w) - \sigma & y_2(w) \\ y_1'(w) & y_2'(w) - \sigma \end{vmatrix} = 0 \quad (5.8)$$

Since $W(w) = 1$, this determinant becomes $\sigma^2 - \beta\sigma + 0 = 0$ (5.9)
where $\beta = y_1(w) + y_2'(w)$

Let σ_1 and σ_1 be the roots of equation (5.9); corresponding to each of these roots, we can find a set of constants c_1 and c_2. By using these constants, we can obtain the solutions, say, S_1 $Z_1(t)$ and $Z_2(t)$, of (8.1) such that

$$\left.\begin{array}{l} Z_1(t+w)=\sigma_1 Z_1(t) \\ Z_1(t+w)=\sigma_2 Z_2(t) \end{array}\right\} \qquad (5.10)$$

Let $\phi(t)$ be a fundamental matrix of the two - dimensional system corresponding to (8.1) with $\phi(0)=1$.

Then, (See [10]), there exists a constant matrix C such that

$$\phi(t+w)=\phi(t)C \qquad (5.11)$$

Moreover there is a constance matrix R such that

$$C=e^{wR} \qquad (5.12)$$

The characteristics roots of R are called the characteristics exponents of (5.1). From the relation 5.11, it is clear that $\phi(w)=C$ and the characteristics roots σ_1 and σ_2 of C are given by (5.9). Now let λ_1 and λ_2 be the characteristic exponents of (5.1). From relation 5.12 we obtain

$$\sigma_1 = \exp.(w\lambda_i) \quad i=1,2: \qquad (5.13)$$

Further, the characteristic roots σ_1 and σ_2 satisfy $\sigma_1\ \sigma_2 = 1$. Hence if $|\beta| = 2$, then (5.9) has a double root. If we consider $\beta = 2$ the double root is $\sigma=1$. Similarly, when $\beta=-2$ the double root is $\sigma=-1$. Thus from (5.5) we have $y(t+w)=y(t)$ for $\sigma = 1$

$$y(t+2w)=-1y(t+w)=y(t) \text{ for } \sigma = -1 \qquad (5.14)$$

This implies that the exists a solution of (5.1) with period w if $\beta = 2$ and with the period 2w if $\beta=-2$.

6.0 Conclusion

Under the assumption of Theorem (*) and (**), the analytic solutions of the system (3.5) was established and it was discovered that the system (5.2) admits at least one solution $y=y(t)$ having the property (3.5) provided that the polynomial equation of the fundamental matrix $\phi(t)$ has no root lying on the imaginary axis, strictly speaking the roots should have negative real part for the stability of $\phi(\lambda)$.

Furthermore, we also established that the solution $y(t+w)=y(t)+1 \ \ \forall t$ exist for the system (3.5)

The differential equations with periodic coefficients considered in this paper arise in three main ways. In some practical problems they occur naturally because some factors in the problem itself periodic such as Hill's equation solved as example in this paper. Another type of problem is that in which we have to find a solution of a partial differential equation where the solution has to be such as to satisfy given boundary conditions at certain special surfaces, in particular elliptic cylinders.

A third source of periodic differential equations is of mainly mathematical interest. Many mathematician, not without reasons, have come to regard ODE has been directed mostly on existence - theorems and similar results for equations of general type. Only rarely does one find mention, especially at postgraduate level, of any problems in connection with process of actually solving such equations. The electric computer may perhaps be partly to blame for this. Since the impression prevails in many quarters that almost differential equation problems can be rarely "put in the machine" so that finding an analytic solution is largely a waste of time. This however is only a small part of the truth for at the higher levels there are generally so many parameters or boundary conditions involved that numerical solutions, even if practicable give no real ideal of the properties of the equation.

It is in this perspective that in this paper we have sought to give an account of important equations and the special function which they generate such as Matheus equation. The paper tried to keep on eye on the physical origin of the equations and be mindful of their significance, in the material world, of their solutions.

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