Journal of the Nigerian Association of Mathematical Physics
Volume 10 (November 2006), **385 - 390**

Numerical integrators for Stiff and Stiff oscillatory First Order initial value problems

V. A. Aladeselu,
Department of Computer Science, University of Benin, Benin City
Nigeria.

Abstract

In this paper, efforts are geared towards the numerical solution of the first order initial value problem (I.V.P) of the form $Y^{/} = F(X,Y)$, $X \in [a, b]$, $Y(a) = Y_0$, where $Y^{/}$ is the total derivative of Y with respect to X.. The scheme so developed for the stated equation is in the same line of thought as Fatunla (1980). It is of order 6, L-stable and exponentially fitted.

1.0 Introduction

Consider a first order initial value problem of the form

$$Y' = F(X,Y),\ Y(a) = Y_o\ ,\ X \in [a, b] \qquad (1.1)$$

which could be linear or non-linear.

***Definition* 1.1**

An equation involving a relation between an unknown function and one or more of its derivatives is called a differential equation and a differential equation together with a prescribed initial condition is called an initial value problem

Systems of ordinary differential equations whose Jacobian, $J = \delta f/\delta y = \delta y'/\delta y$, have at least one eigen value with a very large negative real part characterize stiff systems while those with a very large imaginay part characterize highly oscillatory systems.

In order to minimize errors of computation, researchers have found ways into the development of numerical integrators for solving stiff, stiff oscillatory and oscillatory systems of ordinary differential equations of the form (1.1).

Problems with solution components containing widely separated time independent variable scales are said to be stiff problems asserts Fatunla (1988). Albeit stiff Initial Value Problems were first encountered in the study of the motion of springs of varying stiffness. They are of frequent occurrence in the mathematical formulation of physical situations in control theory and mass action kinectics where processes with widely varying time constants are usually encountered. Curtis and Hirschfelder (1952) were the first proponents of numerical integrators which are well suited to stiff I.V.Ps. Several numerical integrators have since then been developed to solve the problems of interest we are here investigating. Prominent among these are : Fatunla (1976, 1978,1980, 1982, 1998, 1989, 1990, 1991,1993) ; Lambert (1973, 1974,1976, 1978) ; Ascher and Matthies (1998) ; Dahlquist (1963) ; Hall (1982, 1986) ;Prothero (1976) ; Skeel (1985) ; Cryer (1973) ; Lee and Praiser (1978) ; Novikor, V.A and Novikor, E.A (1987) ; Robertson (1976) ; William (1982) ; Willoughby (1974).

***Definition* 1.2** (Fatunla 1988)

Let α be the smallest in absolute value of the real part of all the eigen values of the solution of the initial value problem (1.1) in the form

$$Y(x) = \sum C_j e^{\lambda j\, x} Z_j + \phi(x) \qquad (1.2)$$

and satisfying the relation $Re(\lambda_j) < 0\ \forall j$. If there exists a real no $x^* > 0$ such that $e^{\alpha x^*} \to 0$, then

(i) the transient phase is the interval $(0, x^*)$;
(ii) (x^*, ∞) is called the **stiff phase**;
(iii) $x = x^*$ is called the **transient point**.

Fatunla asserts " In the trasient (i.e non-stiff) phase , stability does not pose any serious problem but stepsize h is chosen so as to resolve the rapid change in order to cope with the required accuracy. The non-stiff code is more efficient than the stiff code which is more expensive per integration step in this zone". He further asserted that the computation of x^* is not easy but a good estimate for it can be obtained $\Leftrightarrow e^{\alpha x^*} = \text{TOL}$ where $\alpha = max(Re\lambda_j) < 0$) and TOL (allowance error tolerance) is specified.

***Remark* 1.1**

Non-stiff algorithms have a definite region of absolute stability while stiff algorithms have unbounded region of absolute stability and this accounts for why stiff algorithms accommodate the use of a large meshsize, *h*, outside the transient or non-stiff phase**.**

For the treatment of stiff problems, the Jacobian matrix $J = \delta f/\delta y = \delta y'/\delta y$ plays a prominent/deciding role. No wonder, Lambert (1974), in his defense for new stiff numerical integrators asserted that the real computational difficulty with stiff systems is centered round the need to repeatedly compute matrix inverses.

The research findings of Fatunla, his predecessors and colleagues clearly indicate that research work in this area is unlimited. This was the motivating factor for the present work which is an extension of Fatunla (1980)[4] "numerical integrations for stiff and highly oscillatory differential equations". It is of order 4.

In this paper, we develop a new numerical integrator of order 6, by introducing a new but real stiffness/oscillatory parameter Λ_2, a diagonal matrix, into the interpolation function (1.7) of Fatunla (1980), which is of order 4. It thus gives a more accurate result than Fatunla (1980). Recall that the higher the order of a numerical integrator the more accurate is the scheme.

2.0 Development of Scheme

Consider the I.V.P $$y' = f(x, y),\ y(0) = y_o; \quad x\ \varepsilon\ [a, b] \qquad (2.1)$$

On every subinterval $I_n = [x_n, x_{n+1}]$. Let the theoretical solution $y(x)$ of (2.1) be approximated by the interpolating function:

$$F(x) = (I - e^{\Lambda 1x})A + (I - e^{\Lambda 1^* x})A^* + e^{\Lambda 2x}B + C \qquad (2.2)$$

where A,B,C are *m*-tuples with complex entries and (*) denotes complex conjugate. Λ_1,Λ_2 are diagonal (Stiffness/Oscillation) matrices with $\Lambda_1$complex and Λ_2 real [Fatunla used the interpolating function $F(x) = (I - e^{\Lambda x})A + (I - e^{\Lambda^* x})B + C$]

If y_n denotes the numerical approximation to the theoretical solution $y(x_n)$ at $x = x_n$ and let $f_n = f(x_n,y_n)$ then the following constraints are imposed on (2.2.)

***Constraint* 2.1**

Equation (2.2) coincides with the theoretical solution at the endpoints of the subinterval

=> $$Y_{n+j} = F(x_{n+j}) \text{ for } j = 0,1 \qquad (2.3^A)$$

Constraint 2.1

The first derivative of the equation (2.2) coincides with the R.H.S of the equation (2.1) at the left endpoint of I_n

$\Rightarrow$ $$f_n = F'(x_n) \qquad (2.3^B)$$

From constraint 2.1,

$$y_n = F(x_n) = (I - e^{\Lambda 1xn})A + (I - e^{\Lambda 1^*xn})A^* + e^{\Lambda 2xn}B + C,\ j = 0$$

$$y_{n+1} = F(x_{n+1}) = (I - e^{\Lambda 1xn+1})A + (I - e^{\Lambda 1^*xn+1})A^* + e^{\Lambda 2xn+1}B + C \qquad : j = 1$$

Since $x_{n+1} = x_n + h$ and $\Delta \mathbf{y}_n = Y_{n+1} - Y_n$, then

$$\Delta y_n = [(I - e^{\Lambda 1h})\, e^{\Lambda 1xn}A + (I - e^{\Lambda 1^*h})\, e^{\Lambda 1^*xn}A^* + (I - e^{\Lambda 2h})e^{\Lambda 2xn}B \qquad (2.4)$$

From constraint 2.2,

$$f_n = F'(x_n) = -\Lambda_1 e^{\Lambda 1xn}A - \Lambda_1^* e^{\Lambda 1^*xn}A^* + \Lambda_2\, e^{\Lambda 2xn}B \qquad (2.5)$$

=> $$f_n^{(1)} = (-\Lambda_1^{\ 2})e^{\Lambda 1xn}A + (-\Lambda_1^{*2})e^{\Lambda 1^*xn}A^* + (\Lambda_2^{\ 2})e^{\Lambda 2xn}B \qquad (2.6)$$

$$f_n^{(2)} = (-\Lambda_1^{\ 3})e^{\Lambda 1xn}A + (-\Lambda_1^{*3})e^{\Lambda 1^*xn}A^* + (\Lambda_2^{\ 3})e^{\Lambda 2xn}B \qquad (2..7)$$

solving equations (2.5) – (2.7) for $e^{\Lambda 1xn}A$, $e^{\Lambda 1^*xn}A^*$, $e^{\Lambda 2xn}B$, the following results were obtained, using Cramer's rule ($D \neq 0$), where

$D = \Lambda_1^2\Lambda_2^2\Lambda_1^*(\Lambda_1 - \Lambda_2) + \Lambda_1\Lambda_1^{*2}\Lambda_2^2(\Lambda_2 - \Lambda_1^*) + \Lambda_2\Lambda_1^2\Lambda_1^{*2}(\Lambda_1^* - \Lambda_1)$.

$$e^{\Lambda 1xn}A = [-\Lambda_1^*\Lambda_2(\Lambda_2 - \Lambda_1^*)(\Lambda_1^*\Lambda_2)f_n - (\Lambda_1^* + \Lambda_2)\, f_n^{(1)} + f_n^{(2)}\ ]/D \qquad (2.8)$$

$e^{\square 1*xn}A^* = [\ \Lambda_1\Lambda_2(\Lambda_2 - \Lambda_1)(\Lambda_1\Lambda_2)\ f_n\ -\ (\Lambda_1 + \Lambda_2)\ f_n^{(1)} + f_n^{(2)}\ ]/D$ (2.9)

$e^{\square 2xn}B\ =\ [\ - \Lambda_1\Lambda_1^*(\Lambda_1 - \Lambda_1^*)(\Lambda_1^*\Lambda_1)f_n - (\Lambda_1^* + \Lambda_1)f_n^{(2)}\ ]\ /\ D$. (2.10)

Inserting equations (2.8), (2.9), (2.10), in equation (2.4), the following result is obtained.

$$y_{n+1}\ =\ y_n\ + Rf_n + Sf_n^{(1)} + Tf_n^{(2)} \qquad (2.11)$$

where

$$R = [(\Lambda_1^*\Lambda_2)^2 (\Lambda_1^* - \Lambda_2) (I - e^{\square 1h}) + (\Lambda_1\Lambda_2)^2(\Lambda_2 - \Lambda_1) (I - e^{\square 1*h})$$
$$+ (\Lambda_1\Lambda_1^*)^2(\Lambda_1 - \Lambda_1^*)(I - e^{\square 2h})\ ]/D \qquad (2.12^A)$$

$$S = [(\Lambda_1^*\Lambda_2)(\Lambda_2^2 - \Lambda_1^{*2}) (I - e^{\square 1h}) - (\Lambda_1\Lambda_2)(\Lambda_2^2 - \Lambda_1^2) (I - e^{\square 1*h})$$
$$+ \Lambda_1\Lambda_1^*(\Lambda_1^{*2} - \Lambda_1^2)(I - e^{\square 2h})]/D(2.12B) \qquad (2.12^B)$$

$$T = [\Lambda_1^*\Lambda_2(\Lambda_1^* - \Lambda_2)(I - e^{\square 1h})\ +\ (\Lambda_1\Lambda_2)(\Lambda_2 - \Lambda_1) (I - e^{\square 1*h})$$
$$+ \Lambda_1^*\Lambda_1((\Lambda_1 - \Lambda_1^*)(I - e^{\square 2h})\ ]/D \qquad (2.12^C)$$

For real constants λ, u, let $\Lambda_1 = \lambda + iu$. Then $\Lambda_1^* = \lambda - iu$ and if $\Lambda_2 = \alpha$, then

$$D =\ - 2ui\alpha\ (\ \lambda^2 + u^2\ )(\ \lambda^2 + u^2 + \alpha^2 - 2\alpha\ \lambda) \qquad (2.13)$$

$$R = 2\alpha^2\ i[e^{\lambda h}((3u\lambda^2\ - 2u\alpha\lambda - u^3)\mathrm{Cos}(hu) -\ (\lambda^3 - \alpha\lambda^2 - 3u^2\lambda + \alpha\ u^2)\mathrm{Sin}(hu))$$
$$- (3u\lambda^2 - 2u\alpha - u^3)\ + (u\ /\ \alpha^2\ )\ (\ \lambda^2 + u^2\ )^2(1 - e^{\alpha h})]/D \qquad (2.14^A)$$

$$S =\ - 2\alpha\ i[e^{\lambda h}((3u\lambda^2\ - u\alpha^2 - u^3\ )\mathrm{Cos}(hu) + (\alpha^2\ \lambda - \lambda^3 + 3u^2\lambda\ )\mathrm{Sin}(hu))$$
$$- (3u\lambda^2\ - u\alpha^2 - u^3)\ \ + (2u\lambda/\alpha\ )(\lambda^2 + u^2\ )^2(1 - e^{\alpha h})\ ]\ /D \qquad (2.14^B)$$

$$T =\ - 2\alpha i[e^{\lambda h}((-2u\lambda + u\alpha)\mathrm{Cos}(hu) + (\lambda^2\ - \alpha\lambda - u^2)\mathrm{Sin}(hu)) - (-2u\lambda + u\alpha\ )$$
$$- (u\ /\alpha\ )(\ \lambda^2 + u^2\ )^2(1 - e^{\alpha h})]/D \qquad (2.14^C)$$

When λ, u, α are simultaneously close to zero, results 2.14A –2.14C, respectively reduce to R(λ, u, α) = h; S(λ, u, α) = $h^2/2 = h^2/2!$; T(λ, u, α) = $h^3/6 = h^3/3!$ and the resultant numerical integrator 2.11 reduces to the form

$$y_{n+1} =\ y_n + hf_n + h^2/2!f_n^{(1)} + h^3/3!f_n^{(2)},$$

which is the popular Taylor series algorithm of order 3.

3.0 Illustrative examples

3.1 Consider the system of initial value problem given in the matrix form

$$Y' = \begin{pmatrix} -10^{-5} & 10^2 & 0 \\ -10^2 & -10^{-5} & 0 \\ 0 & 0 & -3 \end{pmatrix} y \ \text{ such that } \ y(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$y = f(x),\ \ x = 0(\pi)10\pi,$ when solved simultaneously, componentwisely, the following results are obtained.

$$y(x) = [y_1(x),\ y_2(x),\ \ y_3(x)]^T = \left[e^{-10x^5}\sin 100x,\ \ e^{-0x^{-5}}\cos 100x,\ \ e^{-3x}\right]^T$$

Putting $\lambda = -10^{-5} = -0.00001$, $u = 100$, $\alpha = -3$, $h = \pi$, the following tables were obtained

x	Theoretical solution	Scheme 2.11(New scheme)	Fatunla
0	0	0	0
π	0.1260996162891388	0.1260973663444949	0.1260995965255349
2π	0.2501780688762665	0.2501767179778298	0.2501758211271342
3π	0.3703112602233887	0.3702508136156766	0.3702531962086348
4π	0.4844122529029846	0.4844025119369659	0.4844052137569985
5π	0.5907797217369080	0.5908155483783548	0.5908192173469040
6π	0.6877176165580750	0.6877981272298164	0.6877977485145640
7π	0.7736803889274597	0.7738003121029644	0.7737995162133360
8π	0.8472975492477417	0.8474498824787009	0.8474498749136380
9π	0.9073958396911621	0.9075735639928455	0.9075736957869616
10π	0.9530172348022461	0.9532123111640780	0.9532134964569742

Table 3.11: For first component $y_1(x)$

x	Theoretical solution	Scheme 2.11(New scheme)	Fatunla
0	1	1	1
π	0.9919859170913696	0.9919859300085195	0.9919859185503837
2π	0.9681348800659180	0.9681351695535401	0.9681349636037512
3π	0.9288063049316406	0.9288296841684505	0.9288291047738156
4π	0.8746961355209351	0.874691591904627 7	0.8746900538198805
5π	0.8066381812095642	0.8065981489172787	0.8065975206438138
6π	0.7257184982299805	0.7256391963260711	0.7256366003532041
7π	0.6332288384437561	0.6331018108939643	0.6331023180485358
8π	0.5306450128555298	0.5304680767972223	0.5304662731520197
9π	0.4196038842201233	0.4193751775021828	0.4193715663883311
10π	0.3018770217895508	0.3015915809405487	0.3015901821144804

Table 3.12: For second component $y_2(x)$

x	Theoretical solution	Scheme 2.11(New scheme)	Fatunla
0	1	1	1
π	8.039396198000759D-05	8.048672079973838D-05	8.018317126529234D-05
2π	6.463189095029520D-09	9.922946466605448D-08	9.920506111642726D-08
3π	5.196016685492244D-13	9.276679577168541D-08	9.276679380978634D-08
4π	4.177280622238511D-17	9.276627621185596D-08	9.276627621169823D-08
5π	3.358306031041626D-21	9.276627617008651D-08	9.276627617008650D-08
6π	2.699873243371485D-25	9.276627617008316D-08	9.276627617008316D-08
7π	2.170545014962843D-29	9.276627617008316D-08	9.276627617008316D-08
8π	1.744976731744553D-33	9.276627617008316D-08	9.276627617008316D-08
9π	1.402869824370772D-37	9.276627617008316D-08	9.276627617008316D-08
10π	0	9.276627617008316D-08	9.276627617008316D-08

Table 3.13: For Third component $y_3(x)$

As a second illustrative example, consider the moderately stiff system, as per Shampine (1975), given by the following initial value problem:

$$Y' = \begin{pmatrix} -0.1 & -199.9 \\ 0 & -200 \end{pmatrix} y$$

such that $y(0) = (2,1)^T$, $x \in [0,\ 0.01]$. The theoretical solution is given by relation,

$$y(x) = \begin{pmatrix} e^{-200x} + e^{-0.1} \\ e^{-200x} \end{pmatrix}$$

The following results were obtained with h = 0.001, R = h, S = 0.5h^2 and T= (1/6)h^3

x	Theoretical solution	Scheme 2.11(New scheme)	Fatunla
0	2	2	2
0.001	1.818630695343018	1.818566659031208	1.819899992474861
0.002	1.670120000839233	1.670003486466152	1.671095127556475
0.003	1.548511624336243	1.548352140577421	1.549245900651371
0.004	1.448929071426392	1.448734335561482	1.449466084455360
0.005	1.367379665374756	1.367156052587995	1.367755157970899
0.006	1.300594329833984	1.300347295414058	1.300837801371310
0.007	1.245897054672241	1.245630810156932	1.246032402440160
0.008	1.201096892356873	1.200814640435255	1.201143436379126
0.009	1.164399385452271	1.164104153952370	1.164373349349644
0.010	1.134335756301880	1.134030049606273	1.134250448124508

Table 3.21: For first component $y_1(x)$

x	Theoretical solution	Scheme 2.11(New scheme)	Fatunla
0	1	1	1
0.001	0.8187307715415955	0.8186666592800975	0.8199999927237514
0.002	0.6703200340270096	0.6702034726817048	0.6712951137720276
0.003	0.5488116145133972	0.5486520975008471	0.5495458575747971
0.004	0.4493289887905121	0.4491342555633859	0.4498660044572644
0.005	0.3678794503211975	0.3676559280389826	0.3682550334218862
0.006	0.3011941909790039	0.3009471186847341	0.3014376246419854
0.007	0.2465969324111939	0.2463305698032041	0.2467321620864320
0.008	0.2018965333700180	0.2016143250130302	0.201931209569016
0.009	0.1652988791465759	0.1650037520175571	0.1652729474148305
0.010	0.1353352814912796	0.1350295535294757	0.1352469520427115

Table 3.22: For second component $y_2(x)$

4.0 Order and local truncation error

Let V(y(x)) be the operator associated with the integration formula 2.11(new scheme), where $V(y(x)) = y(x + h) - y(x)$ arbitrary function $y(x)$ ε $c^7(s)$.

Then the local truncation error T_{n+1} at $x = x_{n+1}$ is given by V(y(x), h) where $y(x_n)$ is assumed to be the theoretical solution to the initial value problem 1.1. By using the Taylor expansion of $V(y(x),h)$ about $x = x_n$, with the localizing assumption that there is no previous error (i.e $y_n = y(x_n)$), the truncation error for the integration formula 2.11 is derived thus.

$$V_{n+1} = y(x_{n+1}) - y_{n+1} = y(x_n + h) - y_n - Rf_n - Sf_n^{(1)}Tf_n^{(2)} \cdots \quad (4.1)$$

$$= y(x) + hy^{(i)}(x_n) + \frac{h^2}{2!}y^{(ii)}(x_n) + \frac{h^3}{3!}y^{(iii)}(x_n) + \frac{h^4}{4!}y^{(iv)}(x_n) + \frac{h^5}{5!}y^{(v)}(x_n) + \frac{h^6}{6!}y^{(vi)}(x_n)$$

$$+ \frac{h^7}{7!}y^{(vii)}(x_n) - y_n - Rf_n - Sf_n^{(i)} - Tf_n^{(2)} + 0(h^8)$$

$$= \frac{h^7}{7!}f_n^{(6)} - Rf_n - Sf_n^{(i)} - Tf_n^{(2)} + 0(h^8) \quad (4.2)$$

{as the coefficients of h^1, h^2, h^3, h^4, h^5, h^6, vanish, $y_n = y(x_n)$}. Result 4.2 shows that scheme 2.11 is of order 6.

5.0 Stability considerations

To solve stiff systems, numerical integrators possessing special stability properties such as A-stability, *L*-stability and A_o–stability are required.

***Definition* 5.1** (Fatunla 1988)

A one-step numerical integrator is said to be A-stable, if when applied to the scalar test equation $y' = \lambda y$, where λ is a complex constant such that $Re(\lambda) < 0$, the resultant numerical solution is of the form $y_{n+1} = \mu(\lambda h)y_n$, where the stability polynomial $\mu(\lambda h)$ satisfies the inequality $|\mu(\lambda h)| < 1 \ \forall \ Re(\lambda h) < 0$.

***Definition* 5.2** (Fatunla 1980)

A one-step numerical integrator is said to be *L*-stable, if apart from being A-stable, when applied to the scalar test equation $y' = \lambda y$, where λ is a complex constant such that $Re(\lambda) < 0$, the resultant numerical solution is of the form $y_{n+1} = \mu(\lambda h)y'_n$, where the characteristic equation $\mu(\lambda h)$ is such that $lim\ |\mu(\lambda h)| = 0$, as $Re(\lambda h) \to -\infty$.

***Definition* 5.3** (Fatunla 1980)

A numerical integrator scheme is said to be exponentially fitted at a complex value $\lambda = \lambda_o$, if when applied to the initial value problem 1.1, with exact initial condition, the characteristic equation $\mu(\lambda h)$ satisfies the relation $\mu(\lambda_o h) = e^{\lambda_o h}$

$y' = \lambda y$ (scalar test equation)

=> $y'_n = \lambda y_n$ => $f_n = y'_n = \lambda y_n$ => $f_n^{(1)} = \lambda y_n^{1} = \lambda(\lambda y_n) = \lambda^2 y_n$,*and* $f_n^{(2)} = \lambda^3 y_n$

⇨ $y_{n+1} = y_n + R\lambda y_n + S\lambda^2 y_n + T\lambda^3 y_n = \mu(\lambda h)y_n$

⇨ where $\mu(\lambda h) = 1 + \lambda R + \lambda^2 S + \lambda^3 T$.

⇨ $|\mu(\lambda h)| < 1 \ \forall \ Re(\lambda h) < 0$.

⇨ Thus scheme 2.11 is *A*-stable.

Stiffness parameter imaginary => $\lambda = 0$ =>

$R(0,u,\alpha,h) = \{-2\alpha^2/2u^3\alpha(u^2 + \alpha^2)\}[- u^3\cos(hu) - \alpha u^2 \sin(hu) + u^3 + (u^5 / \alpha^2)(1-e^{\alpha h})]$

$= 1/\{u^3(u^2 + \alpha^2)\}[(\alpha u^3\cos(hu) + \alpha^2u^2 \sin(hu) - u^3) - u^5(-h-\alpha h^2 - \alpha^2h^3 + ..)]$

$= h + 0(h)$ when $\alpha = 0$

=>$R(0,u,0,h) = h$.

$S(0,u,\alpha,h) = 1/\{u^3(u^2 + \alpha^2)\}[u(1 - \cos(hu))(+ \alpha^2 + u^2] = (1/ u^2)(1 - \cos(hu))$ whenever $\alpha = 0$.

=>$S(0,u,0,h) = (1 - \cos(hu))/ u^2$

$T(0,u,\alpha,h) = 1/\{u^3(u^2 + \alpha^2)\}[u\alpha\cos(hu) - u^2\sin(hu) - u\alpha + u^3h]$

$= (-1/u^3)((\sin(hu) - uh)$, whenever $\alpha = 0$.

=> $T(0,u,0,h) = (-1/u^3)((\sin(hu) - uh)$

$\therefore \quad \mu(\lambda h) = 1 + \lambda h + \lambda^2/u^2 (1 - \cos(hu)) - \lambda^3/u^3(\sin(hu) - uh))$

$= 1 + iuh - 1 + \cos(hu) + i\sin(hu) - iuh = \cos(hu) + i\sin(hu) = e^{iuh}$ whenever $\lambda = iu$.

=>$\mu(\lambda h) = e^{iuh} = e^{(iu)h} = e^{\lambda h}$.

This shows that the scheme has an exponential fitness.

Also $\lim \mu(\lambda h) = 0$ as $Re(\lambda h) \to -\infty$. This confirms the *L*-stability of the scheme as it has been shown earlier that it is A-stable.

6.0 Conclusion

In this paper, we developed a numerical integrator of order 6, L-stable and exponentially fitted. It compares favourably with existing methods to address the problem of interest. In particular, the scheme performs better than Fatunla (1980) [4] as evidenced in the two illustrating examples as the integration progresses.

References

[1]. Cryer C.W.,(1973): "A new class of highly stable methods: A-stable methods", BIT 13, pp.153-159.

[2] Curtis, C.F. & Hirschfelder, J.O.,(1952), "Integration of stiff equations", Pro.Nat.Acad.Science, U.S., vol. 38, pp235-243

[3] Dahlquist, G., (1963): "A special stability problem for linear multistep methods", BIT3 , pp27-43.

[4] Fatunla, S.O.,(1980), "Numerical integrators for stiff and highly oscillatory differential equations",Maths.Comp., vol.34, No 150, pp373-390

[5] Fatunla,S.O., (1988), "Numerical methods for initial value problems in ordinary differential equations",Academic press, Inc, San Diego.

[6] Lambert, J.D., (1973), "Computational methods in ordinary differential equations",John Wiley and Sons, New York.

[7] Willoughby, R., (1974), "Stiff differential system", Wildbad, W. Germany, Plenum Press, New York.