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On the dynamic buckling of lightly damped cylindrical shells modulated by a periodic load

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Abstract

The dynamic buckling load of finite imperfect, lightly but viscously damped cylindrical shells subjected to a periodic load, is determined using the technique of multiple scaling (two-timing) regular perturbation analysis. The geometric imperfection, assumed deterministic, are also assumed small and are expanded in a double Fourier series. The dynamic buckling load is obtained asymptotically and the result is found to be implicit in the load parameter.

1.0 Introduction

The dynamic buckling analyses of cylindrical shells under various load conditions have been a subject of intense studies for some time now. However, most of these early studies dwelt extensively on loading histories that did not contain the time variable explicitly. For example, Amazigo and Lockhart [1] analysed the step loading dynamic buckling of externally pressurized imperfect cylindrical shells. We recall that the subject of dynamic buckling was originally developed by Budiansky and Hutchinson [2-4] from the pioneering static buckling theory earlier developed by Koiter [5,6]. In [5,6], the geometric imperfections were assumed to be in the shape of the classical buckling mode. However, Lockhart and Amazigo [1] neglected such assumption but rather represented the imperfection in the form of a double Fourier series. They showed, among other things, that the dynamic buckling load depends on the first term in the Fourier sine series representation of the imperfection. We shall exploit this finding.

Perhaps a major feature distinguishing this investigation from most similar ones till date is that the loading history is explicit in the time variable. This renders the Mathematical formulation non-autonomous. In this case, the usual "make –shift" simple technique such as phase plane analysis, developed by Budiansky and Hutchinson [2-4] for analysing autonomous systems, are inappropriate. Similar analyses have been treated by Wang and Tian [7-9], Aksogan and Sofiyev [10], Heinen and Bullesbach [11], Zhu et al [12] and popov [13], among others.

2.0 Karman-Donnell Equations

The necessary standard Karman-Donnell equations governing the deformation of a lightly damped imperfect right circular cylindrical shell are given in terms of the displacement W (X, Y, T) and Airy stress functions F(X, Y, T), where X, Y and T are the axial coordinate, circumferential variable and the time variable respectively .The relevant compatibility and equilibrium equations are respectively given [1] by

$$\frac{1}{Eh}\nabla^4 F - \frac{1}{R}W_{,XX} = -S\left(W, \frac{1}{2}W + \overline{W}\right) \tag{2.1}$$

$$\rho W_{,TT} + \beta W_{,T} + D\nabla^4 W + \frac{1}{R} F_{,XX} = S\left(W + \overline{W}, F\right) - \overline{p}(T) \qquad (2.2)$$

Where the cylindrical shell is of length L, radius R, thickness h, bending stiffness D= $\frac{Eh^3}{12\left(1-\nu^2\right)}$, E is the Young's modulus and ν is the Poisson's ratio, ρ is the mass per unit area, β is the viscous damping constant, $\overline{p}(T)$ is the time dependent external periodic pressure and $\overline{W}(X,Y)$ is a time independent stress-free twice differentiable initial normal displacement. An alphabetic subscript following a comma indicates partial differentiation and S is symmetric bilinear operator defined by

$$S(P,Q) = P_{,XX} Q_{,YY} + P_{,YY} Q_{,XX} - 2P_{,XY} Q_{,XY} \qquad (2.3)$$

While ∇^4 is the biharmonic operator namely $\nabla^4 = \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2$. We now introduce the following nondimensional quantities:

$$x = \frac{\pi X}{L}, \ y = \frac{Y}{R}, \in \overline{w} = \frac{\overline{W}}{h}, \ t = \frac{T\pi^2 \left(\frac{D}{\rho}\right)^{\frac{1}{2}}}{L^2}, \ \lambda \cos\phi\, t = \frac{L^2 R\, \overline{p}}{\pi^2 D}, \qquad (2.4a)$$

$$A = \frac{L^2 \sqrt{12\left(1-\nu^2\right)}}{\pi^2 RL}$$

$$\xi = \frac{L^2}{\pi^2 R^2}, \ K(\xi) = -\frac{A^2}{(1+\xi)^2}, \ H = \frac{h}{R}, \ 2\in = \frac{L^2 \beta}{\pi^2 (D\rho)^{\frac{1}{2}}} \qquad (2.4b)$$

We shall consider homogeneous initial conditions and shall neglect both the axial and circumferential inertia. We shall assume simply supported boundary conditions and neglect boundary layer effects by assuming that the pre-buckling deflection is constant. We now write

$$F = \frac{1}{2}\overline{p}(T) R\left(X^2 + \frac{\alpha}{2} Y^2\right) + \frac{Eh^2 L^2}{\pi^2 R (1+\xi)^2} f \ ; \ W = \frac{\overline{p} R^2 \left(1 - \frac{\alpha\nu}{2}\right)}{Eh} + hw \qquad (2.5)$$

The first terms on the right hand sides of (2.5) for F and W, represent the pre-buckling approximations, while the parameter α takes the value α =1 if the pressure contributes to axial stress through end plates otherwise α =0 if pressure acts laterally . If we substitute (42.a) and (2.5) into (2.1) and (2.2) and simplify, we get

$$\overline{\nabla}^4 f - (1+\xi)^2 w_{,xx} = -H(1+\xi)^2 \overline{S}\left(w, \frac{w}{2} + \in \overline{w}\right) \qquad (2..6)$$

$$w_{,tt} + 2\in w_{,t} + \overline{\nabla}^4 w - K(\xi) f_{,xx} + \overline{\lambda}\left[\frac{\alpha}{2}(w + \in \overline{w})_{,xx} + \xi (w + \in \overline{w})_{,yy}\right] \cos\phi\, \tilde{t}$$

$$= -HK(\xi)\overline{S}(w + \in \overline{w}, f) \qquad (2.7)$$

where

$$\overline{\nabla}^4 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \ ; \ \overline{S}(P,Q) \equiv P_{,xx} Q_{,yy} + P_{,yy} Q_{,xx} - 2P_{,xy} Q_{,xy} \qquad (2.8)$$

The simply supported boundary conditions imply

$$w = w_{,xx} = f = f_{,xx} = 0 \text{ at } x = 0, \pi \qquad (2.9)$$

while the homogeneous initial conditions imply

$$w = w_{,t} = 0 \text{ at } t = 0,\ 0 < x < \pi \ ,0 < y < 2\pi \qquad (2.10)$$

We note that $\overline{\lambda}$ is a nondimensional load parameter while $\in$ is the amplitude of the imperfection .We shall assume $|\in| << 1$ and ϕ is such that $0 < \phi < 2\pi$. We define the dynamic buckling load λ_D as the maximum value of $\overline{\lambda}$ for which the problem (2.6)-(2.10) has bounded solutions (*w*,*f*) for all time t>0 .

3.0 Classical Buckling load λ_C

This is defined as the minimum value of $\overline{\lambda}$ for which there exists a nontrivial solution to the and corresponding linear static problem of the perfect cylindrical shells .The associated compatibility equilibrium equations are respectively given by

$$\overline{\nabla}^4 f - (1+\xi)^2 w_{,xx} = 0 \qquad (3.1a)$$

with boundary conditions

$$\overline{\nabla}_w^4 - K(\xi)F_{,xx} + \overline{\lambda}\left[\frac{\alpha}{2}(\omega + \varepsilon\overline{\omega})_{,xx} + \xi(\omega + \varepsilon\overline{\omega})_{,xx}\right] = 0 \qquad (3.1b)$$

$$w = w_{,\text{xx}} = f = f_{,xx} = 0 \text{ at } x = 0, \pi \qquad (3.1c)$$

The solution to (3.1a,b) is a superposition of the form

$$(w, f)(a_{mk}, b_{mk})\sin(ky + \phi_{mk})\sin mx\,, (a_{mk}, b_{mk}) \neq (0,0) \qquad (3.2a)$$

On substituting (3.2a) into (3.1a,b) and simplifying, we get

$$\overline{\lambda} = \left[\frac{(m^2 + k^2\xi)^2 - \dfrac{m^2(1+\xi)^2 K(\xi)}{(m^2 + k^2\xi)^2}}{\dfrac{\alpha m^2}{2} + k^2\xi}\right];\ \text{m} = 1,2,3;\cdots;\text{k} = 0,1,2,3;\cdots \qquad (3.2b)$$

On maximizing $\overline{\lambda}$ with respect to k, assuming that *k* varies continuously, we get

$$\overline{\lambda} = \left[\frac{(1+\zeta)^2 - \dfrac{(1+\xi)^2 K(\xi)}{(1+\zeta)^2}}{\dfrac{\alpha}{2} + \zeta}\right];\ \zeta = \text{n}^2\xi \qquad (3.3a)$$

where n , in (3.3a), is a critical integer value of k that minimizes $\overline{\lambda}$. Based on above results, we now have

$$(w, f) = \left[\left\{1, \left(\frac{1+\xi}{1+\zeta}\right)^2\right\}\{\, a_{1n} \sin m(ny + \phi_{1n})\,\}\sin x\right] \qquad (3.3b)$$

4.0 Dynamic case, periodic loading

We now embark on the full solution of (2.6)-(2.10), bearing in mind that equation (2.7) has (a) a damping term given proportional to the first degree of the velocity and (b) a sinusoidal term $\cos\phi t$. We let

$$\overline{w} = \overline{a}\sin x \sin ny\ ,\ |\,\overline{\text{a}}\,| < 1 \qquad (4.1)$$

and recall [2,4] that if V_a should be the maximum displacement as a function of the load parameter $\overline{\lambda}$,then , the condition for dynamic buckling is

$$\frac{d\overline{\lambda}}{dV_a} = 0. \qquad (4.2)$$

In order that condition (4.2) be evoked, our first pre-occupation will be to determine a uniformly valid asymptotic expression of the effective maximum displacement V_a subsequent upon which condition (4.2) is used to determine the dynamic buckling load λ_D which normally satisfies the inequality $0 < \lambda_D < \lambda_S < \lambda_C$, where λ_S is the static buckling load. We let

$$\lambda = \frac{\bar{\lambda}}{\lambda_C}, \delta = \lambda \in, \ \tau = \delta t \ , \ 0 < \delta << 1 \tag{4.3}$$

The original compatibility and equilibrium equations (2.6, 2.7) now take the forms

$$\bar{\nabla}^4 f - (1+\xi)^2 w_{,xx} = -H(1+\xi)^2 \bar{S}\left(w, \frac{w}{2} + \frac{\delta \bar{w}}{\lambda} \right) \tag{4.4}$$

$$w_{,tt} + \frac{2\delta w_{,t}}{\lambda} + \bar{\nabla}^4 w - K(\xi) f_{,xx} + \delta\lambda_C \left[\frac{\alpha}{2}\left(\frac{w}{\in} + \bar{w} \right)_{,xx} + \xi\left(\frac{w}{\in} + \bar{w} \right)_{,yy} \right] \cos\phi t$$

$$= -HK(\xi)\bar{S}\left(w + \frac{\delta \bar{w}}{\lambda}, f \right) \tag{4.5}$$

The displacement *w*(*x*,*y*,*t*) can now be considered as a function of the spatial coordinates *x*, and *y* as well as of the time variables t and τ and so we now write $w(x, y, t) = V(x, y, t, \tau; \delta, \in)$ .Thus we have

$$w_{,t} = V_{,t} + \delta V_{,\tau} ; \quad w_{,tt} = V_{,tt} + 2\delta V_{,t\tau} + \delta^2 V_{,\tau\tau} \tag{4.6}$$

We let

$$\begin{pmatrix} f(x, y, t, \tau) \\ V(x, y, t, \tau) \end{pmatrix} = \sum_{i=1}^{\infty} \begin{pmatrix} f^{(i)}(x, y, t, \tau) \\ V^{(i)}(x, y, t, \tau) \end{pmatrix} \delta^i \tag{4.7}$$

We substitute (4.6) into (4.5), using (4.7), and equate coefficients of powers of δ and obtain the following sequence of equations:

$$N\left(f^{(1)}, V^{(1)}\right) \equiv \bar{\nabla}^4 f^{(1)} - (1+\xi)^2 V^{(1)}_{,xx} = 0 \tag{4.8a}$$

$$M\left(f^{(1)}, V^{(1)}\right) \equiv V^{(1)}_{,tt} + \bar{\nabla}^4 V^{(1)} - K(\xi) f^{(1)}_{,xx} = -\lambda_C \left(\frac{\alpha \bar{w}_{,xx}}{2} + \xi \bar{w}_{,yy} \right) \cos\phi t \tag{4.8b}$$

$$N\left(f^{(2)}, V^{(2)}\right) = -H(1+\xi)^2 \left\{ \frac{1}{2} \bar{S}\left(V^{(1)}, V^{(1)}\right) + \frac{1}{\lambda} \bar{S}\left(V^{(1)}, \bar{w}\right) \right\} \tag{4.9a}$$

$$M\left(f^{(2)}, V^{(2)}\right) = -\frac{\lambda_C}{\in}\left(\frac{\alpha V^{(1)}_{,xx}}{2} + \xi V^{(1)}_{,yy} \right) \cos\phi t - HK\left[\bar{S}\left(f^{(1)}, V^{(1)}\right) + \frac{1}{\lambda} \bar{S}\left(f^{(1)}, \bar{w}\right) \right]$$

$$- \frac{2 V^{(1)}_{,t}}{\lambda} - 2 V^{(1)}_{,t\tau} \tag{4.9b}$$

$$N\left(f^{(3)}, V^{(3)}\right) = -H(1+\xi)^2 \left\{ \bar{S}\left(V^{(1)}, V^{(2)}\right) + \frac{1}{\lambda} \bar{S}\left(V^{(2)}, \bar{w}\right) \right\} \tag{4.10a}$$

$$M\left(f^{(3)}, V^{(3)}\right) = -\frac{\lambda_C}{\in}\left(\frac{\alpha V^{(2)}_{,xx}}{2} + \xi V^{(2)}_{,yy} \right) \cos\phi t - HK\left[\bar{S}\left(f^{(2)}, V^{(1)}\right) \right.$$

(4.10b

$$\left. + \frac{1}{\lambda} \bar{S}\left(f^{(2)}, \bar{w}\right) + \bar{S}\left(f^{(1)}, V^{(2)}\right) \right] - 2V^{(2)}_{,t\tau} - V^{(1)}_{,,\tau\tau} - \frac{2}{\lambda}\left(V^{(2)}_{,t} + V^{(1)}_{,\tau} \right)$$

$$N\left(f^{(4)},V^{(4)}\right)=-H(1+\xi)^2\left\{\bar{S}\left(V^{(1)},V^{(3)}\right)+\bar{S}\left(V^{(2)},V^{(2)}\right)+\frac{1}{\lambda}\bar{S}\left(V^{(3)},\bar{w}\right)\right\} \quad (4.11a)$$

$$M\left(f^{(4)},V^{(4)}\right)=-\frac{\lambda_C}{\in}\left(\frac{\alpha V^{(3)}{}_{,xx}}{2}+\xi V^{(3)}{}_{,yy}\right)\cos\phi\, t - HK\left[\frac{1}{\lambda}\bar{S}\left(f^{(3)},\bar{w}\right)+\bar{S}\left(f^{(3)},V^{(1)}\right)\right.$$

$$\left.+\bar{S}\left(f^{(2)},V^{(2)}\right)+\bar{S}\left(f^{(1)},V^{(3)}\right)\right]-2V^{(3)}_{,t\tau}-V^{(2)}_{,\tau\tau}-\frac{2}{\lambda}\left(V^{(3)}_{,t}+V^{(2)}_{,\tau}\right) \quad (4.11b)$$

The boundary conditions are

$$V^{(i)}=V^{(i)}_{,xx}=f^{(i)}=f^{(i)}_{,xx}=0\ at\ x=0,\pi,\ i=1,2,3,\cdots \quad (4.12)$$

while the initial conditions , evaluated at $(t,\tau)=(0,0)$, are

$$V^{(i)}=0,\ V^{(1)}_{,t}=0,\ \ V^{(k)}_{,t}+V^{(r)}_{,\tau}=0\ ,\ \ i=1,2,3,\cdots;\ k=r+1;\ r=1,2,3,\cdots \quad (4.13)$$

4.1 Solution of equations of first order of perturbation

We let

$$\begin{pmatrix} f^{(i)}(x,y,t,\tau) \\ V^{(i)}(x,y,t,\tau)\end{pmatrix}=\sum_{p,q=1,2,3,\cdots}^{\infty}\left\{\begin{pmatrix} f^{(i)}_{1r}(x,y,t,\tau) \\ V^{(i)}_{1r}(x,y,t,\tau)\end{pmatrix}\cos qy+\begin{pmatrix} f^{(i)}_{2r}(x,y,t,\tau) \\ V^{(i)}_{2r}(x,y,t,\tau)\end{pmatrix}sinqy\right\}\sin px \quad (4.14)$$

Because of the nature of (4.1) (to be substituted into (4.8b) later), we are convinced that when $i=1$,we shall have $f^{(1)}_{11}=V^{(1)}_{11}\equiv 0$. Thus we let

$$\begin{pmatrix} f^{(1)}(x,y,t,\tau) \\ V^{(1)}(x,y,t,\tau)\end{pmatrix}=\sum_{p,q=1,2,3,\cdots}^{\infty}\begin{pmatrix} f^{(i)}_{2r}(x,y,t,\tau) \\ V^{(i)}_{2r}(x,y,t,\tau)\end{pmatrix}sinqy\sin px \quad (4.15)$$

The subscript *r* in (4.14) and (4.15) denotes, for each i, r-multiples of n-circumferential waves, as for example *in*, $\sin rny\sin mx$ .All through in this analysis, any integration with respect to x has 0 and π as the lower and upper limits respectively while integration with respect to y has 0 and 2π as the lower and upper limits of integration respectively .If we substitute (4.15) into (4.8a) and thereafter, multiply by $\sin rny\sin mx$,we get, for $q = rn,\ p = m$

$$f^{(1)}_{2r}=-\frac{(1+\xi)^2 m^2 V^{(1)}_{2r}}{\left\{m^2+(rn)^2\xi\right\}^2} \quad (4.16a)$$

In particular, for *r* = 1, we have

$$f^{(1)}_{21}=-\frac{(1+\xi)^2 m^2 V^{(1)}_{21}}{\left\{m^2+n^2\xi\right\}^2} \quad (4.16b)$$

We next substitute (4.15) into (4.8b) and get

$$V^{(1)}_{21,tt}+\Omega^2 V^{(1)}_{21}=\bar{a}\lambda_C\left(\frac{\alpha}{2}+n^2\xi\right)\cos\phi\, t \quad (4.17a)$$

$$V^{(1)}_{21}(,0,0)=0\ ,\ V^{(1)}_{21,t}(0,0)=0 \quad (4.17b)$$

$$\Omega^2_r=\left\{\left(m^2+(rn)^2\xi\right)\right\}^2+\left(\frac{m^2A}{\left(m^2+(rn)^2\xi\right)}\right)^2; \quad (4.17c)$$

$$\Omega_1\equiv\Omega,\ \Omega_r\neq r\phi\ ,r\Omega_r\neq\phi,\ r=1,2,3,\cdots$$

On solving (4.17a,b), we get

$$V_{21}^{(1)}(t,\tau)=a_1(\tau)\cos\Omega\, t+b_1(\tau)\sin\Omega\, t+\frac{\bar{a}\lambda_C}{\Omega^2-\phi^2}\left(\frac{\alpha}{2}+n^2\xi\right)\cos\phi\, t \qquad (4.18a)$$

$$a_1(0)=-B\,,\ \mathrm{B}=\frac{\bar{a}\lambda_C}{\Omega^2-\phi^2}\left(\frac{\alpha}{2}+n^2\xi\right)\ ,\ \mathrm{b}_1(0)=0 \qquad (4.18b)$$

4.2: Solution of second order perturbation equations

If we substitute on the right hand sides of (4.9a,b) and simplify, we get

$$N\left(f^{(2)},V^{(2)}\right)=n^2(1+\xi)^2H\left[\frac{1}{2}\left(V_{21}^{(1)}\right)^2+\frac{\bar{a}}{\lambda}V_{21}^{(1)}\right](\cos 2\mathrm{x}+\cos 2\mathrm{ny}) \qquad (4.19a)$$

$$M\left(f^{(2)},V^{(2)}\right)=\frac{\lambda}{\in}\left(\frac{\alpha}{2}+n^2\xi\right)\mathrm{V}_{21}^{(1)}\sin x\sin ny\cos(\phi\, t)-2\mathrm{V}_{21,t\,\tau}^{(1)}\sin x\sin ny$$

$$+HK(\xi)n^2\left[V_{21}^{(1)}f_{21}^{(1)}+\bar{a}f_{21}^{(1)}\right](\cos 2\mathrm{x}+\cos 2\mathrm{ny})-\frac{2\mathrm{V}_{21,t}^{(1)}}{\lambda}\sin x\sin ny \qquad (4.19b)$$

$$V^{(2)}(x,y,0,0)=0\ ,\ \mathrm{V}_{,\mathrm{t}}^{(2)}(x,y,0,0)+\mathrm{V}_{,\tau}^{(1)}(x,y,0,0)=0 \qquad (4.19c)$$

To solve (4.19a-c), we substitute (4.14) into (4.19a), for $i=2$, and multiply both sides by cosrnysinmx and note that for $p = m$, $q = nr$ with $r = 2$, we get

$$f_{12}^{(2)}=\frac{4n^2(1+\xi)^2H}{m\pi\left(m^2+4n^2\xi\right)^2}\left[\frac{1}{2}\left(V_{21}^{(1)}\right)^2+\frac{\bar{a}}{\lambda}V_{21}^{(1)}\right]-\frac{m^2(1+\xi)^2V_{12}^{(2)}}{\left(m^2+4n^2\xi\right)^2},\ \mathrm{m\ odd} \qquad (4.20a)$$

Similarly, we substitute 4.14) into (4.19a), for $i=2$, and multiply both sides by *sinrny sinmx* and see that for $p = m$, $q = nr$, we get

$$f_{2r}^{(2)}=-\frac{m^2(1+\xi)^2V_{2r}^{(2)}}{\left\{m^2+(nr)^2\xi\right\}^2}\quad\forall\mathrm{r} \qquad (4.20b)$$

We next substitute (4.14) into (4.19b), for $i=2$, and multiply both sides by *cosrny sinmx* , and get ,using (4.20a)

$$V_{12,tt}^{(2)}+\Omega_2^2V_{12}^{(2)}=Q_i^{(m)}\left(V_{21}^{(1)}\right)^2+\bar{a}Q_2^{(m)}V_{21}^{(1)} \qquad (4.21a)$$

$$V_{12}^{(2)}(0,0)=0\ ,\ V_{12,\mathrm{t}}^{(2)}(0,0)=0 \qquad (4.21b)$$

$$\mathrm{Q}_1^{(\mathrm{m})}=\frac{4n^2A^2H}{m\pi}\left\{\frac{m^2}{2\left(m^2+4n^2\xi\right)}+\frac{1}{(1+n^2\xi)^2}\right\},$$

$$\mathrm{Q}_2^{(\mathrm{m})}=\frac{4n^2A^2H}{m\pi}\left\{\frac{m^2}{\lambda\left(m^2+4n^2\xi\right)}+\frac{1}{(1+n^2\xi)^2}\right\} \qquad (4.21c)$$

Similarly, by multiplying (4.19b) by sinrny sinmx, we get, using (4.20b) , where *p = m = 1, q = nr*, for *r = 1*

$$V_{21,tt}^{(2)}+\Omega_2^2V_{21}^{(2)}=\frac{\lambda_C}{\in}\left(\frac{\alpha}{2}+n^2\xi\right)\left\{a_1\cos\Omega\,\mathrm{t}+\mathrm{b}_1\sin\Omega\,\mathrm{t}\right\}\cos\phi\,\mathrm{t}+2\Omega\left(\mathrm{a}_1'\cos\Omega\,\mathrm{t}-\mathrm{b}_1'\sin\Omega\,\mathrm{t}\right)$$

$$+\frac{2\Omega}{\lambda}\left(\mathrm{a}_1\cos\Omega\,\mathrm{t}-\mathrm{b}_1\sin\Omega\,\mathrm{t}\right) \qquad (4.22a)$$

$$V_{21}^{(2)}(0,0)=0\ ,\quad V_{21,\,\mathrm{t}}^{(2)}(0,0)+V_{21,\,\tau}^{(1)}(0,0)=0 \qquad (4.22b)$$

where $\frac{d(\ )}{d\tau}=(\ )'$ .It is our intention in this analysis , to limit our investigation to the case where the displacement $V(x,y,t,\tau)$ is in the shape of the classical buckling mode. We shall hence neglect the solution of (4.21a-c) since such a solution will definitely yield the result $\sum_{m=1,3,5,\cdots}^{\infty} V_{12}^{(2)}(t,\tau)\cos 2ny \sin mx$ which is not in the shape of the classical buckling mode .In later analysis, we may however re-instate this neglected term where such re-instatement ensures clarity of thought and makes the analysis more explicit. To ensure a uniformly valid solution in the time scale t, we equate to zero in (44.22a) the coefficients of $\cos\Omega\,\mathrm{t}$ and $\sin\Omega\,\mathrm{t}$ and get respectively

$$b_1' + \frac{b_1}{\lambda} = 0\ ,\quad \text{and}\quad a_1' + \frac{a_1}{\lambda} = 0 \tag{4.23a}$$

The solutions of (4.23a) , using (4.18b), are

$$b_1(\tau) \equiv 0\,,\ a_1(\tau) = -Be^{-\frac{\tau}{\lambda}}\ ,\ a_1'(0) = \frac{B}{\lambda}, a_1''(0) = -\frac{B}{\lambda^2} \tag{4.23b}$$

Thus, we have

$$V_{21}^{(1)}(t,\tau) = a_1 \cos\Omega\,\mathrm{t} + \mathrm{B}\cos\phi\,\mathrm{t} \tag{4.23c}$$

The remaining equation in (35a,b) is solved to yield

$$V_{21}^{(2)}(t,\tau) = a_2(\tau)\cos\Omega\,\mathrm{t} + \mathrm{b}_2(\tau)\sin\Omega\,\mathrm{t} + \frac{\mathrm{q}_0}{2}\left[\frac{B}{\Omega^2} + \frac{B\cos 2\phi\,\mathrm{t}}{\Omega^2 - 4\phi^2} + \frac{a_1}{\phi}\left\{\frac{\cos(\Omega-\phi)\,\mathrm{t}}{2\Omega - \phi} - \frac{\cos(\Omega+\phi)\,\mathrm{t}}{2\Omega+\phi}\right\}\right] \tag{4.24a}$$

$$a_2(0) = -\frac{Bq_0S_0}{2}\,,\quad \mathrm{q}_0 = \frac{\lambda_{\mathrm{C}}}{\in}\left(\frac{\alpha}{2} + n^2\xi\right),\ \mathrm{b}_2(0) = \frac{B}{\Omega\lambda} \tag{4.24b}$$

$$S_0 = \left[\frac{1}{\Omega^2} + \frac{1}{\Omega^2 - 4\phi^2} - \frac{1}{\phi}\left\{\frac{1}{2\Omega - \phi} - \frac{1}{2\Omega+\phi}\right\}\right] \tag{4.24c}$$

In the next round of substitution, we shall need the following simplifications

$$\left(V_{21}^{(1)}\right)^3 = (a_1\cos\Omega\,\mathrm{r} + \mathrm{B}\cos\phi\,\mathrm{t})^3 = R_1\cos\Omega\,\mathrm{t} + \mathrm{R}_2\cos 3\Omega\,\mathrm{t} + \mathrm{R}_3\cos\phi\,\mathrm{t} + \mathrm{R}_4\cos 3\phi\,\mathrm{t}$$
$$+\mathrm{R}_5\{\cos(2\Omega+\phi)\,\mathrm{t} + \cos(2\Omega - \phi)\,\mathrm{t}\} + R_6\{\cos(\Omega+2\phi)\,\mathrm{t} + \cos(\Omega - 2\phi)\,\mathrm{t}\} \tag{4.25a}$$

where $R_1(\tau) = 3\left(\frac{a_1^3}{4} + \frac{a_1B^2}{2}\right)$, $\mathrm{R}_1(0) = -\frac{9B^3}{4}$; $\mathrm{R}_2(\tau) = \frac{a_1^3}{4}$, $\mathrm{R}_2(0) = -\frac{B^3}{4}$, (4.25b)

$$R_3 = 3\left(\frac{a_1^2B}{2} + \frac{B^3}{4}\right),\ R_3(0) = \frac{9B^3}{4};\quad \mathrm{R}_4(\tau) = \mathrm{R}_4(0) = \frac{B^3}{4}\ ;$$

$$\mathrm{R}_5(\tau) = \frac{3a_1^2B}{4},\ \mathrm{R}_5(0) = \frac{3B^3}{4} \tag{4.25c}$$

$$R_6(\tau) = \frac{3a_1B^2}{4},\ R_6(0) = -\frac{3B^2}{4} \tag{4.25d}$$

We shall also need

$$\left(V_{21}^{(1)}\right)^2 = (a_1\cos\Omega\,\mathrm{t} + \mathrm{B}\cos\phi\,\mathrm{t})^2 = R_7 + R_8\cos 2\Omega\,\mathrm{t} + \mathrm{R}_9\cos 2\phi\,\mathrm{t}$$
$$+\mathrm{R}_0\{\cos(\Omega+\phi)\,\mathrm{t} + \cos(\Omega-\phi)\,\mathrm{t}\} \tag{4.26a}$$

where

$$R_7(\tau)=\frac{1}{2}\left(a_1^2+b_1^2\right),\ \mathrm{R}_7(0)=\mathrm{B}^2\ ;\ \mathrm{R}_8(\tau)=\frac{a_1^2}{2},\ \mathrm{R}_8(0)=\frac{B^2}{2},$$
$$\mathrm{R}_{10}(\tau)=a_1B,\ \mathrm{R}_{10}(0)=-B^2 \tag{4.26b}$$

4.3 Solution of third order perturbation equations

We now substitute for terms on the right hand sides of (4.10a,b),and simplify to get

$$N\left(f^{(3)},V^{(3)}\right)=-(1+\xi)^2H\left[-n^2\left\{V_{21}^{(1)}V_{21}^{(2)}+\frac{\bar{a}}{\lambda}V_{21}^{(2)}\right\}(\cos 2\mathrm{ny}+\cos 2\mathrm{x})\right.$$
$$+\sum_{m=1,3,5,\cdots}^{\infty}\left\{V_{21}^{(1)}V_{12}^{(2)}+\frac{\bar{a}}{\lambda}V_{12}^{(2)}\right\}\left\{\left(4n^2+m^2n^2\right)\cos 2ny\sin ny\sin \mathrm{x}\ \sin \mathrm{mx}\right.$$
$$\left.\left.+4n^2\sin 2ny\cos ny\cos x\sin \mathrm{mx}\right\}\right] \tag{4.27a}$$

$$M\left(f^{(3)},V^{(3)}\right)=\left[\frac{\lambda_C}{\in}\left(\frac{\alpha}{2}+n^2\xi\right)V_{21}^{(2)}\cos\phi\,\mathrm{t}-2V_{21,\mathrm{t}\tau}^{(2)}-V_{21,\tau\tau}^{(1)}\right]\sin x\sin \mathrm{ny}$$
$$+\sum_{\mathrm{m}=1,3,5,\cdots}^{\infty}\left\{\frac{\lambda_C}{\in}\left(\frac{\alpha\,\mathrm{m}^2}{2}+4n^2\xi\right)V_{12}^{(2)}\cos\phi\,\mathrm{t}--2V_{12,\mathrm{t}\tau}^{(2)}\right\}\cos 2ny\sin \mathrm{mx}$$
$$-\mathrm{HK}(\xi)\left[\left(\frac{\bar{a}}{\lambda}f_{12}^{(2)}+V_{21}^{(1)}f_{12}^{(2)}-Q_7V_{21}^{(1)}V_{21}^{(2)}\right)\left\{\left\{-\mathrm{n}^2(\cos 2ny+\cos 2x)\right.\right.\right.$$
$$\left.\left.\left.+\sum_{m=1,3,5,\cdots}^{\infty}\left\{\left(4n^2+m^2n^2\right)\cos 2ny\sin ny\sin x\sin mx+4n^2m\sin 2ny\cos nyy\cos x\cos mx\right\}\right\}\right\}\right]$$
$$-\frac{2}{\lambda}\left[V_{21,\mathrm{t}}^{(2)}\sin ny\sin x+\sum_{m=1,3,5,\cdots}^{\infty}V_{12,\mathrm{t}}^{(2)}\cos 2ny\sin mx+V_{21,\tau}^{(1)}\sin ny\sin x\right] \tag{4.27b}$$

$$V^{(3)}(x,y,0,0)=0\ ,\ \mathrm{V}_{,\mathrm{t}}^{(3)}(x,y,0,0)+V_{,\tau}^{(2)}(x,y,0,0)=0 \tag{4.27c}$$

where
$$Q_7=\left(\frac{1+\xi}{1+n^2\xi}\right)^2 \tag{4.27d}$$

We substitute (4.14), for $i=3$, into (4.27aa) ,and multiply both sides by *cosrny sinmx* and note that for $q = rn$, r =2 and $p = m$, we get

$$f_{12}^{(3)}=\frac{4n^2H(1+\xi)^2}{\pi\, m\left(m^2+4n^2\xi\right)^2}\left(V_{21}^{(1)}V_{21}^{(2)}+\frac{\bar{a}}{\lambda}V_{21}^{(2)}\right)-\frac{(1+\xi)^2m^2V_{12}^{(2)}}{\left(m^2+4n^2\xi\right)^2},\ \mathrm{m\ \ odd} \tag{4.28a}$$

Similarly, by substituting (4.14), for i= 3, into 4.27a), and multiplying by sinrnysinmx for r = 1 and r = 3 (in each of the two cases) and for $p = m$, we get

$$f_{21}^{(3)}=-\frac{2\mathrm{H}(1+\xi)^2}{\pi^2\left(m^2+n^2\xi\right)^2}\sum_{m=1,3,5,\cdots}^{\infty}\left(V_{21}^{(1)}V_{12}^{(2)}+\frac{\bar{a}}{\lambda}V_{12}^{(2)}\right)\left\{4\mathrm{n}^2mQ_8^{(m)}-\left(4n^2+m^2n^2\right)Q_{10}^{(m)}\right\}$$
$$-\frac{(1+\xi)^2m^2V_{21}^{(3)}}{\left(m^2+n^2\xi\right)^2} \tag{4.28b}$$

and $f_{23}^{(3)} = -\frac{2\mathrm{H}(1+\xi)^2}{\pi^2(m^2+9n^2\xi)^2}\sum_{m=1,3,5,\cdots}^{\infty}\left(V_{21}^{(1)}V_{12}^{(2)}+\frac{\bar{a}}{\lambda}V_{12}^{(2)}\right)\left\{4\mathrm{n}^2 mQ_8^{(m)}+\left(4n^2+m^2n^2\right)Q_{10}^{(m)}\right\}$

$$-\frac{(1+\xi)^2 m^2 V_{23}^{(3)}}{(m^2+9n^2\xi)^2} \tag{4.28c}$$

where $Q_8^{(m)} = \frac{\pi}{4}\left[\frac{1}{2m-1}+\frac{1}{2m+1}\right],\ \mathrm{Q}_{10}^{(\mathrm{m})} = \left[\frac{1}{2m-1}+2-\frac{1}{2m+1}\right]$ (4.28d)

Since the terms $V_{21}^{(1)}, V_{21}^{(2)}$ and $V_{12}^{(2)}$ do not depend on the summation index *m*, we can rewrite (4.21a-c) simply as

$$f_{12}^{(3)} = Q_9^{(m)}\left(V_{21}^{(1)}V_{21}^{(2)}+\frac{\bar{a}}{\lambda}V_{21}^{(2)}\right)-\frac{(1+\xi)^2 V_{12}^{(3)}}{(m^2+4n^2\xi)^2},$$

$$f_{21}^{(3)} = -Q_{11}^{(m)}\left(V_{21}^{(1)}V_{12}^{(2)}+\frac{\bar{a}}{\lambda}V_{12}^{(2)}\right)-\frac{(1+\xi)^2 V_{21}^{(3)}}{(m^2+4n^2\xi)^2} \tag{4.29a}$$

$$f_{23}^{(3)} = -Q_{12}^{(m)}\left(V_{21}^{(1)}V_{12}^{(2)}+\frac{\bar{a}}{\lambda}V_{12}^{(2)}\right)-\frac{(1+\xi)^2 V_{23}^{(3)}}{(m^2+4n^2\xi)^2} \tag{4.29b}$$

where

$$Q_9^{(m)} = \frac{4n^2 H(1+\xi)^2}{\pi\, m(m^2+4n^2\xi)^2},$$

$$\mathrm{Q}_{11}^{(\mathrm{m})} = \frac{2\mathrm{H}(1+\xi)^2}{\pi^2(m^2+n^2\xi)^2}\sum_{\mathrm{m}=1,3,5,\cdots}^{\infty}\left\{4\mathrm{n}^2 mQ_8^{(m)}-\left(4n^2+m^2n^2\right)Q_{10}^{(m)}\right\} \tag{4.29c}$$

$$\mathrm{Q}_{12}^{(\mathrm{m})} = \frac{2\mathrm{H}(1+\xi)^2}{\pi^2(m^2+n^2\xi)^2}\sum_{\mathrm{m}=1,3,5,\cdots}^{\infty}\left\{4\mathrm{n}^2 mQ_8^{(m)}+\left(4n^2+m^2n^2\right)Q_{10}^{(m)}\right\} \tag{4.29d}$$

We next substitute (4.14) for i = 3, into (4.27b), and multiply both sides by *sinrny sinmx* and see that for $q = rn$, r = 1 and $p = m$ (odd), we obtain

$$V_{21,\mathrm{tt}}^{(3)}+\left(m^2+n^2\xi\right)^2 V_{21}^{(3)}+K(\xi)m^2 f_{21}^{(3)} = \left\{q_0 V_{21}^{(2)}\cos\phi\,\mathrm{t}-2V_{21,t\tau}^{(2)}-V_{21,\tau\tau}^{(1)}\right\}$$
$$-HK(\xi)\sum\left[\left\{\frac{\bar{a}}{\lambda}f_{12}^{(2)}+V_{21}^{(1)}f_{12}^{(2)}-Q_7V_{21}^{(1)}V_{12}^{(2)}\right\}\left\{4n^2mQ_8^{(m)}-Q_{10}^{(m)}\left(4n^2+m^2n^2\right)\right\}\right]$$
$$-\frac{2}{\lambda}\left(V_{21,t}^{(2)}+V_{21,\tau}^{(1)}\right) \tag{4.30}$$

Using (4.20a), we note the following simplifications of terms as they appear in (4.30)

$$f_{12}^{(2)}\left(\frac{\bar{a}}{\lambda}+V_{21}^{(1)}\right) = Q_4^{(m)}\left[\frac{1}{2}\left(\frac{\bar{a}}{\lambda}\right)\left(V_{21}^{(1)}\right)^3+\frac{3}{2}\left(\frac{\bar{a}}{\lambda}\left(V_{21}^{(1)}\right)^2+\left(\frac{\bar{a}}{\lambda}\right)^2 V_{21}^{(1)}\right)\right]$$
$$-Q_5^{(m)}V_{12}^{(2)}\left(\frac{\bar{a}}{\lambda}+V_{21}^{(1)}\right) \tag{4.31a}$$

$$Q_4^{(m)} = \frac{4n^2H(1+\xi)^2}{m\,\pi(\mathrm{m}^2+4n^2\xi)^2},\quad \mathrm{Q}_5^{(\mathrm{m})} = \frac{m^2(1+\xi)^2}{(\mathrm{m}^2+4n^2\xi)^2} \tag{4.31b}$$

where we have substituted for $f_{12}^{(2)}$ in (4.31a) from (4.20a). If we substitute for $f_{21}^{(3)}$ from (4.29a) into (4.30) and simplify , using (4.31a), we get

$$V_{21,tt}^{(3)}+\Omega^2 V_{21}^{(3)}=\left\{q_0 V_{21}^{(1)}\cos\phi\, t-2V_{21,tt}^{(1)}-V_{21,\tau\tau}^{(1)}\right\}+Q_{14}^{(m)}V_{21}^{(1)}V_{12}^{(2)}+\frac{\bar{a}}{\lambda}Q_{15}^{(m)}V_{12}^{(2)}$$
$$+\frac{1}{2}Q_{16}^{(m)}\left(V_{21}^{(1)}\right)^3+\frac{3}{2}\left(\frac{\bar{a}}{\lambda}\right)Q_{16}^{(m)}\left(V_{21}^{(1)}\right)^2+\left(\frac{\bar{a}}{\lambda}\right)^2 Q_{16}^{(m)}\left(V_{21}^{(1)}\right) \tag{4.32a}$$

$$V_{21}^{(3)}(0,0)=0\ ,\ \mathrm{V}_{21,t}^{(3)}(0,0)+V_{21,\tau}^{(2)} \tag{4.32b}$$

where

$$Q_{14}^{(m)}=-\left[\left(\frac{mA}{1+\xi}\right)^2 Q_{11}^{(m)}-HK(\xi)\sum_{m=1,3,5,\cdots}^{\infty}\left(Q_5^{(m)}+Q_7\right)\left\{4n^2 mQ_8^{(m)}-\left(4n^2+m^2n^2\right)\right\}\right] \tag{4.32c}$$

$$Q_{15}^{(m)}=HK(\xi)\sum_{m=1,3,5,\cdots}^{\infty}Q_5^{(m)}\left[\left\{4n^2 mQ_8^{(m)}-\left(4n^2+m^2n^2\right)\right\}\right] \tag{4.32d}$$

$$Q_{16}^{(m)}=-HK(\xi)\sum_{m=1,3,5,\cdots}^{\infty}Q_4^{(m)}\left[\left\{4n^2 mQ_8^{(m)}-\left(4n^2+m^2n^2\right)\right\}\right] \tag{4.32e}$$

Henceforth, we shall neglect the term $V_{12}^{(2)}$ as previously indicated. We shall simplify the following term as it appears in (4.32a)

$$q_0 V_{21}^{(2)}\cos\phi\,\mathrm{t}=\frac{\mathrm{q}_0}{2}\left[a_2\left\{\cos(\Omega+\phi)\mathrm{t}+\cos(\Omega-\phi)\mathrm{t}\right\}+b_2\left\{\sin(\Omega+\phi)\mathrm{t}+\sin(\Omega-\phi)\mathrm{t}\right\}\right.$$
$$+\frac{q_0}{2}\left\{\left\{\frac{2B\cos\phi\,\mathrm{t}}{\Omega^2}+\frac{B(\cos 3\phi\,\mathrm{t}+\cos\phi\,\mathrm{t})}{\Omega^2-4\phi^2}+\frac{a_1}{\phi}\left\{\left(\frac{\cos\Omega\,\mathrm{t}+\cos(\Omega-2\phi)\mathrm{t}}{2\Omega-\phi}\right)\right.\right.\right.$$
$$\left.\left.\left.\left.-\left(\frac{\cos\Omega\,\mathrm{t}+\cos(\Omega+2\phi)\mathrm{t}}{2\Omega+\phi}\right)\right\}\right\}\right\}\right] \tag{4.33}$$

We now substitute for relevant terms into (4.32), using (4.25a) and (4.26a) and, to ensure a uniformly valid solution in the time scale *t*, equate to zero the coefficients of cost and sint and get respectively

$$b_2'+\frac{b_2}{\lambda}=h_1(\tau)=\frac{1}{2\Omega}\left[Q_{16}^{(m)}\left(\frac{\bar{a}}{\lambda}\right)^2 a_1-\frac{2a_1'}{\lambda}-a_1''+\frac{1}{2}Q_{16}^{(m)}R_1\frac{a_1 q_0}{2\phi}\left\{\frac{1}{2\Omega-\phi}-\frac{1}{2\Omega+\phi}\right\}\right] \tag{4.34a}$$

and

$$a_2'+\frac{a_2}{\lambda}=0 \tag{4.34b}$$

where $h_1(0)=BS_{22}+B^3S_{23}$; $\mathrm{S}_{22}=-\frac{1}{2\Omega}\left[\left(\frac{\bar{a}}{\lambda}\right)^2 Q_{16}^{(m)}-\frac{1}{\lambda^2}+\frac{2}{\lambda}+\frac{q_0}{2\phi}\left\{\frac{1}{2\Omega-\varphi}-\frac{1}{2\Omega+\phi}\right\}\right]$ (4.34c)

$$S_{23}=-\frac{9Q_{16}^{(m)}}{16\Omega} \tag{4.34d}$$

On solving (4.34a,b) subject to (4.14b), we get

$$b_2(\tau)=e^{-\left(\frac{\tau}{\lambda}\right)}\left[\int_0^\tau e^{\left(\frac{s}{\lambda}\right)}h_1(s)\,ds+\frac{B}{\Omega\lambda}\right],\ \ \mathrm{a}_2(\tau)=a_2(0)e^{-\left(\frac{\tau}{\lambda}\right)} \tag{4.35a}$$

We observe from (4.34a,b) that

$$b_2'(0) = BS_{24} + B^3 S_{23} \ , \ a_2'(0) = \frac{Bq_0 S_0}{2\lambda} \ , \ S_{24} = S_{22} - \frac{1}{\lambda} \qquad (4{,}35b)$$

Meanwhile, the remaining equations in the substitution into (4.32a) are

$$V^{(3)}_{21,tt} + \Omega^2 V^{(3)}_{21} = R_{11} + R_{10}\cos 3\Omega t + R_{12}\cos 2\Omega t + R_{13}\cos\phi t + R_{14}\cos 2\phi t + R_{15}\cos 3\phi t$$
$$+ R_{16}\cos(\Omega+\phi)t + R_{17}\sin(\Omega+\phi)t + R_{18}\cos(\Omega-\phi)t + R_{19}\sin(\Omega-\phi)t + R_{20}\cos(\Omega+2\phi)t$$
$$R_{21}\cos(\Omega-2\phi)t + R_{22}\{\cos(2\Omega+\phi)t + \cos(2\Omega-\phi)t\} + q_1\sin 2\phi t \qquad (4.36a)$$

$$V^{(3)}_{21}(0,0) = 0 \ , \ V^{(3)}_{21,t}(0,0) + V^{(2)}_{21,\tau}(0,0) = 0 \qquad (4.36b)$$

where

$$R_{10} = \frac{1}{2}Q^{(m)}_{16}R_2 \ , R_{10}(0) = -\frac{3}{4}B^3 Q^{(m)}_{16} \ ; \ R_{11} = \frac{3}{2}\left(\frac{\bar{a}}{\lambda}\right)Q^{(m)}_{16}R_7 \ ,$$
$$R_{11}(0) = \frac{3}{2}\left(\frac{\bar{a}}{\lambda}\right)Q^{(m)}_{16}B^2 \qquad (4.36c)$$

$$R_{12} = \frac{3}{2}\left(\frac{\bar{a}}{\lambda}\right)Q^{(m)}_{16}R_8 \ , \ R_{12}(0) = \frac{3}{4}\left(\frac{\bar{a}}{\lambda}\right)Q^{(m)}_{16}B^2 \qquad (4.36d)$$

$$R_{13} = \left[\frac{Bq_0^2}{2}\left\{\frac{1}{\Omega^2} + \frac{1}{\Omega^2 - 4\phi^2}\right\} + BQ^{(m)}_{16}\left(\frac{\bar{a}}{\lambda}\right)^2 + \frac{1}{2}Q^{(m)}_{16}R_3\right] ,$$
$$R_{13} = \frac{BS_1}{2} + \frac{9B^3 Q^{(m)}_{16}}{8} \qquad (4.36e)$$

$$S_1 = q_0^2\left(\frac{1}{\Omega^2} + \frac{1}{\Omega^2 - 4\phi^2} + Q^{(m)}_{16}\left(\frac{\bar{a}}{\lambda}\right)^2\right) \qquad (4.36f)$$

$$R_{14} = \left(\frac{3\bar{a}}{2\lambda}\right)Q^{(m)}_{16}R_9 \ , R_{14}(0) = \frac{3}{2}\left(\frac{\bar{a}}{\lambda}\right)Q^{(m)}_{16}B^2 \ ; R_{15} = \frac{q_0^2}{4(\Omega^2 - 4\phi^2)}$$
$$+ \frac{1}{2}Q^{(m)}_{16}R_2 + \frac{1}{2}Q^{(m)}_{16}R_4 \qquad (4.36g)$$

$$R_{15}(0) = BS_2 + \frac{1}{8}B^3 Q^{(m)}_{16} \ , \ S_2 = \frac{q_0^2}{4(\Omega^2 - 4\phi^2)} \qquad (4.36h)$$

$$R_{16} = \frac{q_0 a_2}{2} + \frac{3\bar{a}Q^{(m)}_{16}R_0}{2\lambda} \ , R_{16}(0) = -\frac{Bq_0^2 S_0}{4} - \frac{3\bar{a}B^2 Q^{(m)}_{16}}{2\lambda} \qquad (4.36i)$$

$$R_{17} = \left[\frac{q_0 b_2}{2} - \frac{q_0(\Omega+\phi)}{\phi(2\Omega+\phi}\left(a_1' + \frac{a_1}{\lambda}\right)\right] , R_{17}(0) = \frac{Bq_0}{2\Omega\lambda} \qquad (4.36j)$$

$$R_{18} = R_{16} = \frac{q_0 a_2}{2} + \frac{3\bar{a}Q^{(m)}_{16}R_0}{2\lambda} , R_{18}(0) = R_{16}(0) = -\frac{Bq_0^2 S_0}{4} - \frac{3\bar{a}B^2 Q^{(m)}_{16}}{2\lambda} \qquad (4.36k)$$

$$R_{19} = \frac{q_0 b_2}{2} + \frac{q_0(\Omega-\phi)}{\phi(2\Omega-\phi)}\left(a_1' + \frac{a_1}{\lambda}\right) \ , R_{19}(0) = R_{17}(0) = \frac{Bq_0}{2\Omega\lambda} \qquad (4.36l)$$

$$R_{20} = \left[\frac{1}{2}Q^{(m)}_{16}R_6 - \left(\frac{q_0}{2}\right)^2\left(\frac{a_1}{\phi(2\Omega-\phi)}\right)\right] , R_{20}(0) = BS_3 - B^3 S_4 \qquad (4.36m)$$

$$S_3 = \left(\frac{q_0}{2}\right)^2\left(\frac{1}{\phi(2\Omega-\phi)}\right), \quad S_4 = \frac{3Q_{16}^{(m)}}{8}; \; R_{21} = \left[\frac{1}{2}Q_{16}^{(m)}R_6 + \left(\frac{q_0}{2}\right)^2\left(\frac{a_1}{\phi(2\Omega-\phi)}\right)\right] \quad (4.36n)$$

$$R_{21}(0) = -\left(BS_3 + B^3S_4\right); \; R_{22} = \frac{1}{2}Q_{16}^{(m)}R_6 \quad ; R_{22}(0) = \frac{1}{8}B^3Q_{16}^{(m)}; \; q_1 = \frac{2\phi q_0 B}{\lambda\left(\Omega^2-4\phi^2\right)} \quad (4.36o)$$

On solving (4.36a,b), we get

$$V_{21}^{(3)}(t,\tau) = a_3(\tau)\cos\Omega\,t + b_3(\tau)\sin\Omega\,t + \frac{R_{11}}{\Omega^2} - \frac{R_{10}\cos3\Omega\,t}{8\Omega^2} - \frac{R_{12}\cos2\Omega\,t}{3\Omega^2} + \frac{R_{13}\cos\phi\,t}{\left(\Omega^2-\phi^2\right)}$$

$$+\frac{R_{14}\cos2\phi\,t}{\left(\Omega^2-4\phi^2\right)} + \frac{R_{15}\cos3\phi\,t}{\left(\Omega^2-9\phi^2\right)} - \frac{1}{\phi(2\Omega+\phi)}\left\{R_{16}\cos(\Omega+\phi)\,t + R_{17}\sin(\Omega+\phi)\,t\right\}$$

$$-\frac{1}{\phi(2\Omega-\phi)}\left\{R_{18}\cos(\Omega-\phi)\,t + R_{19}\sin(\Omega-\phi)\,t\right\} - \frac{R_{20}\cos(\Omega+2\phi)\,t}{4\phi(\Omega+\phi)} + \frac{R_{21}\cos(\Omega-2\phi)\,t}{4\phi(\Omega-\phi)}$$

$$+\frac{q_1\sin2\phi\,t}{\Omega^2-4\phi^2} + R_{22}\left[\frac{\cos(2\Omega-\phi)\,t}{(\phi-\Omega)(3\Omega-\phi)} - \frac{\cos(2\Omega+\phi)\,t}{(\phi+\Omega)(3\Omega+\phi)}\right] \quad (4.37a)$$

$$a_3(0) = BS_5 + \left(\frac{\bar{a}}{\lambda}\right)Q_{16}^{(m)}B^2S_6 + Q_{16}^{(m)}B^3S_7) \quad (4.37b)$$

$$S_5 = \left[\frac{S_1}{4\left(\phi^2-\Omega^2\right)} - \frac{S_2}{\left(\Omega^2-9\phi^2\right)} - \frac{q_0^2S_0}{4\phi(2\Omega+\phi)} + \frac{q_0^2S_0}{4\phi(2\Omega-\phi)} - \frac{S_3}{4\phi(\Omega+\phi)}\right] \quad (4.37c)$$

$$S_6 = -\left[\frac{3}{2\Omega^2} + \frac{3}{2\left(\Omega^2-4\phi^2\right)} + \frac{1}{\phi(2\Omega+\phi)} - \frac{3}{2\phi(2\Omega-\phi)} + \frac{1}{4\Omega^2}\right] \quad (4.37d)$$

$$S_7 = -\left[\frac{3}{32\,\Omega^2} + \frac{9}{8\left(\phi^2-\Omega^2\right)} + \frac{1}{8\left(\Omega^2-9\phi^2\right)} - \frac{S_4}{4\phi(\Omega+\phi)} - \frac{S_4}{4\phi(\Omega-\phi)} \right. \quad (4.37e)$$

$$\left. + \frac{3}{8}\left\{\frac{1}{(\phi-\Omega)(3\Omega-\phi)} - \frac{1}{(\phi+\Omega)(3\Omega+\phi)}\right\}\right]$$

$$b_3(0) = -\frac{Bq_0S_8}{\Omega\lambda}, \; S_8 = \left[\frac{(\Omega-\phi)}{2\phi\Omega(2\Omega-\phi)} - \frac{(\Omega+\phi)}{2\phi\Omega(2\Omega+\phi)} + \left(\frac{2\phi}{\left(\Omega^2-4\phi^2\right)}\right)^2 + \frac{S_0}{2} + \frac{1}{2\phi}\left\{\frac{1}{2\Omega-\phi} - \frac{1}{2\Omega+\phi}\right\}\right] \quad (4.37f)$$

We next substitute (4.14) into (4.27c) (for *i* = 3), delete $V_{12}^{(2)}$ wherever it occurs in the substitution and use $f_{23}^{(3)}$ from (4.28b) and finally obtain the following

$$V_{23,tt}^{(3)} + \Omega_3^2V_{23}^{(3)} = Q_{17}^{(m)}\left[\frac{1}{2}\left(V_{21}^{(1)}\right)^3 + \frac{3\bar{a}}{2\lambda}\left(V_{21}^{(1)}\right)^2 + \left(\frac{\bar{a}}{\lambda}\right)^2V_{21}^{(1)}\right] \quad (4.38a)$$

$$V_{23}^{(3)}(0,0) = 0\,, V_{23,t}^{(3)}(0,0) = 0 \quad (4.38b)$$

where $$Q_{17}^{(m)} = -HK(\xi)\sum_{m=1,3,5,\cdots}^{\infty}Q_4^{(m)}\left\{(4n^2+m^2n^2)Q_{10}^{(m)} + 4n^2mQ_8^{(m)}\right\} \quad (4..38c)$$

If we simplify the right hand side of (4.38a), we get

$$V_{23,tt}^{(3)} + \Omega_3^2V_{23}^{(3)} = Q_{17}^{(m)}\left[\,R_{23} + R_{24}\cos\Omega\,t + R_{25}\cos2\Omega\,t + R_{26}\cos3\Omega\,t + R_{27}\cos\phi\,t\right.$$

$$+R_{28}\cos2\phi\,t + R_{29}\cos3\phi\,t + R_{30}\left\{\cos(\Omega+\phi)\,t + \cos(\Omega-\phi)\,t\right\}$$

$$+R_{31}\{\cos(2\Omega+\phi)t+\cos(2\Omega-\phi)t\}+R_{32}\{\cos(\Omega+2\phi)t+\cos(\Omega-2\phi)t\}] \quad (4.39a)$$

$$V_{23}^{(3)}(0,0)=0 \ , V_{23,t}^{(3)}(0,0)=0 \quad (4.39b)$$

where

$$R_{23}=\frac{3\bar{a}R_7}{2\lambda}, \quad R_{23}(0)=\frac{3\bar{a}B^2}{2\lambda} \ ; \ R_{24}=\frac{R_1}{2}+\left(\frac{\bar{a}}{\lambda}\right)^2 a_1 \ ,$$
$$R_{24}(0)=-\left\{\frac{9B^3}{8}+\left(\frac{\bar{a}}{\lambda}\right)^2 B\right\} \quad (4.39c)$$

$$R_{25}=\frac{3\bar{a}R_8}{2\lambda} \ , \ R_{25}(0)=\frac{3\bar{a}B^2}{2\lambda}=R_{23}(0); R_{26}=\frac{R_2}{2} \ , \ R_{26}(0)=-\frac{B^3}{8} \quad (4.39d)$$

$$R_{27}=\left\{\frac{R_3}{2}+\left(\frac{\bar{a}}{\lambda}\right)^2 B\right\} \ , R_{27}(0)=\left\{\frac{9B^3}{8}+\left(\frac{\bar{a}}{\lambda}\right)^2 B\right\}, R_{28}=\frac{3\bar{a}R_9}{2\lambda} \ ,$$
$$R_{28}(0)=\frac{3\bar{a}B^2}{4\lambda} \quad (4.39e)$$

$$R_{29}=\frac{R_4}{2} \ , \ R_{29}(0)=\frac{B^3}{8} \ ; \ R_{30}=\frac{3\bar{a}R_0}{2\lambda} \ , R_{30}(0)=-\frac{3\bar{a}B^2}{2\lambda} \quad (4.39f)$$

$$R_{31}=\frac{R_5}{2} \ , R_{31}(0)=\frac{3B^3}{8} \ ; \ R_{32}=\frac{R_6}{2} \ , \ R_{32}(0)=-\frac{3B^3}{8} \quad (4.39g)$$

On solving (4.39a,b), we get

$$V_{23}^{(3)}(t,\tau)=a_4(\tau)\cos\Omega_3 t+b_4(\tau)\sin\Omega_3 t+\frac{R_{23}}{\Omega_3^2}+\frac{R_{24}\cos\Omega t}{\Omega_3^2-\Omega^2}+\frac{R_{25}\cos 2\Omega t}{\Omega_3^2-4\Omega^2}+\frac{R_{26}\cos 3\Omega t}{\Omega_3^2-9\Omega^2}$$
$$+\frac{R_{27}\cos\phi t}{\Omega_3^2-\phi^2}+\frac{R_{28}\cos 2\phi t}{\Omega_3^2-4\phi^2}+\frac{R_{29}\cos 3\phi t}{\Omega_3^2-9\phi^2}+R_{30}\left\{\frac{\cos(\Omega+\phi)t}{\Omega_3^2-(\Omega+\phi)^2}+\frac{\cos(\Omega-\phi)t}{\Omega_3^2-(\Omega-\phi)^2}\right\}$$
$$+R_{31}\left\{\frac{\cos(2\Omega+\phi)t}{\Omega_3^2-(2\Omega+\phi)^2}+\frac{\cos(2\Omega-\phi)t}{\Omega_3^2-(2\Omega-\phi)^2}\right\}$$
$$+R_{32}\left\{\frac{\cos(\Omega+2\phi)t}{\Omega_3^2-(\Omega+2\phi)^2}+\frac{\cos(\Omega-2\phi)t}{\Omega_3^2-(\Omega-2\phi)^2}\right\} \quad (4.40a)$$

where $a_4(0)=Q_{17}^{(m)}\left[\left(\frac{\bar{a}}{\lambda}\right)^2 BS_{11}+\left(\frac{\bar{a}}{\lambda}\right)B^2S_9+S_{10}B\mathbf{3}\right], b_4(0)=0$ (4.40b)

$$S_9=-\left[\frac{3}{\Omega_3^2\lambda}+\frac{3}{2(\Omega_3^2-4\Omega^2)}+\frac{3}{4(\Omega_3^2-4\phi^2)}-\frac{3}{2}\left\{\frac{1}{\Omega_3^2-(\Omega+\phi)^2}+\frac{1}{\Omega_3^2-(\Omega-\phi)^2}\right\}\right] \quad (4.40c)$$

$$S_{10}=\left[\frac{9}{8(\Omega_3^2-\Omega^2)}+\frac{1}{8(\Omega_3^2-9\Omega^2)}-\frac{9}{8(\Omega_3^2-\phi^2)}-\frac{1}{8(\Omega_3^2-9\phi^2)}\right.$$
$$\left.-\frac{3}{8}\left\{\frac{1}{(\Omega_3^2-(\Omega+2\phi)^2}+\frac{1}{(\Omega_3^2-(\Omega-2\phi)^2}\right\}\right]; \ S_{11}=\left[\frac{1}{\Omega_3^2-\Omega^2}-\frac{1}{\Omega_3^2-\phi^2}\right] \quad (4.40d)$$

We readily observe that at the level of third order of perturbation, the buckling mode splits into three distinct modes, namely $\sum_{1,3,5,\cdots}^{\infty} V_{21}^{(3)} \sin ny \sin mx$, $\sum_{1,3,5,\cdots}^{\infty} V_{23}^{(3)} \sin 3ny \sin mx$ and $\sum_{1,3,5,\cdots}^{\infty} V_{12}^{(3)} \sin 2ny \sin mx$. The last mentioned is certainly not in the shape of the classical mode and so is neglected. The second, that is, $\sum_{m=1,3,5,\cdots}^{\infty} V_{23}^{(3)} \sin 3ny \sin \mathrm{mx}$, while not strictly in the shape of the classical mode, is nevertheless deemed essential to the buckling process and so ,is retained. Thus, the summary of the displacement derivation so far is

$$V(x,y,t,\tau) = \delta V_{21}^{(1)} \sin ny \sin x + \delta^2 V_{21}^{(2)} \sin ny \sin mx + \delta^3 \sum_{m=1,3,5,\cdots}^{\infty} \left[V_{21}^{(3)} \sin ny + V_{23}^{(3)} \sin 3ny \right] \sin mx + O\left(\delta^4\right) \tag{4.41}$$

4.4 Maximum Displacement V_a

The conditions for maximum displacement are as follows:

$$V_{,\mathrm{x}} = V_{,\mathrm{y}} = 0 \;\; ; \;\; \mathrm{V}_{,\mathrm{t}} + \delta\, \mathrm{V}_{,\tau} = 0 \tag{4.42}$$

We let x_a, y_a, t_a and τ_a be the critical values of x, y, t and τ respectively at maximum displacement and now assume the following series

$$\begin{aligned} t_a &= t_0 + \delta\, \mathrm{t}_1 + \delta^2\, \mathrm{t}_2 + \delta^3\, \mathrm{t}_3 + \cdots ; \tau_a = \delta\, t_a \\ &= \delta \left[t_0 + \delta\, \mathrm{t}_1 + \delta^2\, \mathrm{t}_2 + \delta^3\, \mathrm{t}_3 + \cdots \right] \end{aligned} \tag{4.43}$$

On substituting into the first two of (4.42), we see that, for nontrivial solution, we get

$$x_a = \frac{\pi}{2} \; , \mathrm{y}_{\mathrm{a}} = \frac{\pi}{2n} \tag{4.44}$$

where we have taken the least positive values in (4.44) .We now substitute into the third equation in (4.42), using (4.43) and (4.44) and, to obtain the values of t_0 and t_1 , equate to zero the coefficients of δ and δ^2 and obtain the following respective equations

$$V_{21}^{(1)}(t_0, 0) = 0 \;\; ; \text{ and } \left[\mathrm{t}_1 V_{21,\mathrm{tt}}^{(1)} + V_{21,\mathrm{t}\tau}^{(1)} + V_{21,\mathrm{t}}^{(2)} + V_{21,\tau}^{(1)} \right] \Big|_{(\mathrm{t}_0, 0)} = 0 \tag{4.45a}$$

From the first of (4.45a), we get

$$\phi \sin \phi\, \mathrm{t}_0 - \Omega \sin \Omega\, \mathrm{t}_0 = 0 \tag{4.45b}$$

which determines t_0 uniquely. An approximate value of t_0 is

$$t_0 \cong \sqrt{\frac{6}{\phi^2 + \Omega^2}} \tag{4.45c}$$

From the second of (4.45a), we get

$$t_1 = -\left(\frac{V_{21,\mathrm{t}}^{(2)} + V_{21,\tau}^{(1)} + V_{21,\mathrm{t}\tau}^{(1)}}{V_{21,\mathrm{tt}}^{(1)}} \right) \Bigg|_{(\mathrm{t}_0, 0)} \tag{4.45d}$$

where the terms in (4.45d) are evaluated as follows

$$V_{21,\mathrm{t}}^{(1)}(t_0, 0) = BS_{26} \;\; , \mathrm{S}_{26} = \left[\frac{\Omega q_0 S_0 \sin \Omega\, \mathrm{t}_0}{2} + \frac{\cos \Omega\, \mathrm{t}_0}{\lambda} + \frac{q_0}{2} \left\{ \left\{ -\frac{2\phi \sin 2\phi\, \mathrm{t}_0}{\Omega^2 - 4\phi^2} - \frac{1}{\phi} \left\{ \frac{(\Omega + \phi)\sin(\Omega + \phi)\, \mathrm{t}_0}{2\Omega + \phi} - \frac{(\Omega - \phi)\sin(\Omega - \phi)\, \mathrm{t}_0}{2\Omega - \phi} \right\} \right\} \right\} \right] \tag{4.45e}$$

$$V^{(1)}_{21,\tau}(t_0,0)=\frac{B\cos\Omega\,\mathrm{t}_0}{\lambda}\;;\;\mathrm{V}^{(1)}_{21,\,\mathrm{t}\,\tau}(t_0,0)=-\frac{B\Omega\sin\Omega\,\mathrm{t}_0}{\lambda}, \tag{4.45f}$$

$$\mathrm{V}^{(1)}_{21,\,\mathrm{t\,t}}(t_0,0)=B(\Omega^2\cos\Omega\,\mathrm{t}_0-\phi^2\cos\phi\,\mathrm{t}_0)$$

The maximum displacement $V_a(\lambda)$ is determined by evaluating (4.41) at the critical values of the variables. This yields, using (4.43)-(4.45f),

$$V_a=\Big[\,\delta V^{(1)}_{21}+\delta^2\left(t_0V^{(1)}_{21,\tau}+V^{(2)}_{21}\right)+\delta^3\Big\{\Big\{t_1\left(V^{(2)}_{21,t}+V^{(1)}_{21,\tau}\right)+t_0V^{(2)}_{21,\tau}$$

$$+\frac{1}{2}\left\{t_1^2V^{(1)}_{21,\,\mathrm{t\,t}}+2t_0t_1V^{(1)}_{21,t\,\tau}+t_0^2V^{(1)}_{21,\tau\,\tau}\right\}+\sum_{\mathrm{m}=1,3,5,\cdots}^{\infty}\left(\mathrm{V}^{(3)}_{21}-V^{(3)}_{23}\right)\Big\}\Big\}\Big]\Bigg|_{(\mathrm{t}_0\;0)}+O(\delta^4) \tag{4.46}$$

Some crucial terms in (4.46) are evaluated as follows:

$$V^{(2)}_{21}(t_0,0)=BS_{25}\;, \tag{4.47a}$$

$$\mathrm{S}_{25}=\left[-\frac{q_0S_0\cos\Omega\,\mathrm{t}_0}{2}+\frac{\sin\Omega\,\mathrm{t}_0}{\Omega\lambda}+\frac{q_0}{2}\left\{\left\{\frac{1}{\Omega^2}+\frac{\cos2\phi\,\mathrm{t}_0}{\Omega^2-4\phi^2}-\frac{1}{\phi}\left\{\frac{\cos(\Omega-\phi)t_0}{2\Omega-\phi}-\frac{\cos(\Omega+\phi)t_0}{2\Omega+\phi}\right\}\right\}\right\}\right] \tag{4.47b}$$

$$V^{(2)}_{21,\tau}(t_0,0)=BS_{27}+B^3S_{28} \tag{4.47c}$$

$$S_{27}=\left[\frac{q_0S_0\cos\Omega\,\mathrm{t}_0}{2\lambda}+S_{24}\sin\Omega\,\mathrm{t}_0+\frac{q_0}{2\phi\lambda}\left\{\frac{\cos(\Omega-\phi)t_0}{2\Omega-\phi}-\frac{\cos(\Omega+\phi)t_0}{2\Omega+\phi}\right\}\right] \tag{4.47d}$$

$$,\mathrm{S}_{28}=S_{23}\sin\Omega\,\mathrm{t}_0$$

Similarly, we have $\quad V^{(3)}_{31}(t_0,0)=BS_{17}+\left(\frac{\bar{a}}{\lambda}\right)BQ^{(m)}_{16}S_{18}+Q^{(m)}_{16}B^3S_{19}$ (4.47e)

$$S_{17}=\Bigg[S_5\cos\Omega t_0-\frac{q_0S_8\sin\Omega t_0}{\Omega\lambda}+\frac{S_1\cos\phi\,\mathrm{t}_0}{4(\Omega^2-\phi^2)}+\frac{S_2\cos3\phi\,\mathrm{t}_0}{(\Omega^2-9\phi^2)}$$

$$+\frac{q_0^2S_0\cos(\Omega+\phi)\,\mathrm{t}_0}{4\phi(2\Omega+\phi)}-\frac{q_0^2S_0\cos(\Omega-\phi)\,\mathrm{t}_0}{4\phi(2\Omega-\phi)}+\frac{q_0S_8\sin(\Omega-\phi)\,t_0}{2\Omega\lambda} \tag{4.47f}$$

$$-\frac{S_3\sin(\Omega+2\phi)\,t_0}{4\phi(\Omega+\phi)}-\frac{S_3\sin(\Omega-2\phi)\,t_0}{4\phi(\Omega-\phi)}+\frac{2\phi q_0\sin2\phi\,\mathrm{t}_0}{\lambda(\Omega^2-4\phi^2)}\Bigg]$$

$$S_{18}=\left[S_6\cos\Omega t_0+\frac{3}{2\lambda\Omega}-\frac{\cos2\Omega t_0}{4\Omega^2\lambda}+\frac{3\cos2\phi\,\mathrm{t}_0}{2(\Omega^2-4\phi^2)}+\frac{\cos(\Omega+\phi)\,\mathrm{t}_0}{\phi(2\Omega+\phi)}-\frac{3\cos(\Omega-\phi)\,\mathrm{t}_0}{2\phi(2\Omega-\phi)}\right] \tag{4.47g}$$

We also have, from the right hand side of (4.47e),

$$S_{19}=\Bigg[S_7\cos\Omega\,\mathrm{t}_0+\frac{9\cos\phi\,\mathrm{t}_0}{8(\Omega^2-\phi^2)}+\frac{\cos3\phi\,\mathrm{t}_0}{8(\Omega^2-9\phi^2)}+\frac{S_4\cos(\Omega+2\phi)\,\mathrm{t}_0}{4\phi(\Omega+\phi)}$$

$$-\frac{S_4\cos(\Omega-2\phi)\,\mathrm{t}_0}{4\phi(\Omega-\phi)}+\frac{3}{8}\left\{\frac{\cos(2\Omega-\phi)\,\mathrm{t}_0}{(\phi-\Omega)(3\Omega-\phi)}-\frac{\cos(2\Omega+\phi)\,\mathrm{t}_0}{(\phi+\Omega)(3\Omega+\phi)}\right\}+\frac{3\cos3\Omega\,\mathrm{t}_0}{32\Omega^2}\Bigg] \tag{2.47h}$$

$$V^{(3)}_{23}(t_0,0)=Q^{(m)}_{17}\left[\left(\frac{\bar{a}}{\lambda}\right)^2BS_{20}+\left(\frac{\bar{a}}{\lambda}\right)B^2S_{21}+S_{22}B^3\right] \tag{4.47i}$$

where $\quad S_{20}=\left[+\frac{\cos\phi\,\mathrm{t}_0}{\Omega_3^2-\phi^2}+S_{11}\sin\Omega\;3\mathrm{t}_0-\frac{\cos\Omega\,\mathrm{t}_0}{\Omega_3^2-\Omega^2}\right]$ (4.47j)

$$S_{21}=\left[\frac{3}{2\Omega_3^2}+S_9\sin\Omega_3\ t_0+\frac{3\cos 2\Omega\ t_0}{2\left(\Omega_3^2-4\Omega^2\right)}+\frac{3\cos 2\phi\ t_0}{4\left(\Omega_3^2-4\phi^2\right)}-\frac{3}{2}\left\{\frac{\cos(\Omega+\phi)\,t_0}{\Omega_3^2-(\Omega+\phi)^2}\right\}+\frac{\cos(\Omega-\phi)\,t_0}{\Omega_3^2-(\Omega-\phi)^2}\right] \tag{4.47k}$$

$$S_{22}=\left[\frac{9\cos\phi\ t_0}{8\left(\Omega_3^2-\phi^2\right)}+S_{10}\sin\Omega_3\ t_0-\frac{9\cos\Omega\ t_0}{8\left(\Omega_3^2-\Omega^2\right)}-\frac{\cos 3\Omega\ t_0}{8\left(\Omega_3^2-9\Omega^2\right)}+\frac{\cos 3\phi\ t_0}{8\left(\Omega_3^2-9\phi^2\right)}\right.$$
$$\left.+\frac{3}{8}\left\{\frac{\cos(2\Omega+\phi)\,t_0}{\Omega_3^2-(2\Omega+\phi)^2}+\frac{\cos(2\Omega-\phi)\,t_0}{\Omega_3^2-(2\Omega-\phi)^2}\right\}-\frac{3}{8}\left\{\frac{\cos(\Omega+2\phi)\,t_0}{\Omega_3^2-(\Omega+2\phi)^2}+\frac{\cos(2\Omega-\phi)\,t_0}{\Omega_3^2-(\Omega-2\phi)^2}\right\}\right] \tag{4.47l}$$

On simplifying (4.46), using (4.47a-k), we get

$$V_a=\delta C_1+\delta^2C_2+\delta^3C_3+O\left(\delta^{\,4}\right) \tag{4.48a}$$

where

$$C_1=BQ_1\,,\mathrm{C}_2=\mathrm{BQ}_2\,,\mathrm{C}_3=B^3Q_3 \tag{4.48b}$$

$$Q_1=\cos\phi\ t_0-\cos\Omega\ t_0\ \ ,\mathrm{Q}_2=\left(\frac{t_0\cos\Omega\ t_0}{\lambda}+S_5\right) \tag{4.48c}$$

$$Q_3=\left[B^{-2}\left\{\left\{\left(\frac{\cos\Omega\ t_0}{\lambda}+S_{26}\right)t_1+S_{27}t_0+\frac{1}{2}\left\{t_1^2\left(\Omega^2\cos\Omega\,t_0-\phi^2\cos\phi\ t_0\right)-\frac{2\Omega t_1t_0\sin\Omega\,t_0}{\lambda}-\frac{t_0^2\cos\Omega\,t_0}{\lambda}\right\}\right.\right.\right.$$
$$\left.\left.\sum_{m=1,3,5,\cdots}^{\infty}\left\{S_{17}-Q_{17}^{(m)}\left(\frac{\bar a}{\lambda}\right)^2S_{20}\right\}\sin\left(\frac{m\pi}{2}\right)\right\}\right\}+B^{-1}\left\{\frac{\bar a}{\lambda}\sum_{m=1,3,5,\cdots}^{\infty}\left(S_{18}Q_{16}^{(m)}-S_{21}Q_{17}^{(m)}\right)\sin\left(\frac{m\pi}{2}\right)\right\}$$
$$\left.+\left\{t_0S_{28}+\sum_{m=1,3,5,\cdots}^{\infty}\left(S_{19}Q_{16}^{(m)}-S_{22}Q_{17}^{(m)}\right)\sin\left(\frac{m\pi}{2}\right)\right\}\right] \tag{2.48d}$$

4.5 Dynamic buckling load λ_D

We note from (4.48a-d) that the maximum displacement $V_a(\lambda)$ depends on the load parameter λ whose particular value λ_D at buckling we are seeking. The full application of condition (4.2) is now sought and to achieve this we first have [14] to reverse the series (4.48a) and now write

$$\delta=e_1V_a+e_2V_a^2+e_3V_a^3+\cdots \tag{4.49a}$$

To determine $e_i, \mathrm{i}=1,2,3,\cdots$ in (4.49a), we substitute for $V_a(\lambda)$ from (4.48a-d) into (4.49a) and equate the coefficients of δ, δ^2 and δ^3 etc. and so obtain the following respective values:

$$e_1=\frac{1}{C_1}\ \ ,\mathrm{e}_2=-\frac{C_2}{C_1^3}\,,\ \ \mathrm{e}_3=\frac{2C_2^2-C_1C_3}{C_1^5} \tag{4.49b}$$

We note that, except for e_1, the terms e_2 and e_3 depend on λ. The maximization (4.2) is now easily accomplished through (4.49ba) to yield

$$e_1+2e_2V_{aD}+3e_3V_{aD}^2=0 \tag{4.50a}$$

$$V_{aD}=\frac{1}{3e_e}\left\{-e_2\pm\left(e_2^2-3e_1e_3\right)^{\frac{1}{2}}\right\} \tag{4.50b}$$

We shall however use the negative sign of the two signs resulting from square root (the positive sign having no physical relevance in this case). To simplify (4.50b), we note that where $V_{aD}=V_a(\lambda_D)$. From (63a), we get

$$\left(e_2^2-3e_1e_3\right)^{\frac{1}{2}}=\sqrt{\frac{3C_3}{C_1^5}\left(1-\frac{5C_2^2}{3C_1C_3}\right)}=\frac{Q_4\sqrt{3}}{B}\ \ ;\mathrm{Q}_4=\sqrt{\frac{Q_3}{Q_1^5}\left(1-\frac{5Q_2^2}{3B_1^2Q_1Q_3}\right)} \tag{5.51a}$$

Thus, we have

$$-e_2-\left(e_2^2-3e_1e_3\right)^{\frac{1}{2}}=-\left(e_2^2-3e_1e_3\right)^{\frac{1}{2}}\left[1+\frac{1}{\left(e_2^2-3e_1e_3\right)^{\frac{1}{2}}}\right]$$

$$=-\frac{Q_5\sqrt{3}}{B}\ ,\ \mathrm{Q}_5=Q_4\left[1-\frac{Q_2}{\sqrt{3}BQ_1^3Q_4}\right] \qquad (4.51b)$$

From (4.51b), we have

$$e_3=\frac{2C_2^2-C_1C_3}{C_1^5}=-\frac{C_3}{C_1^4}\left(1-\frac{2C_2^2}{C_1C_3}\right)=-\frac{Q_3Q_6}{BQ_1^4};\ \mathrm{Q}_6=\left(1-\frac{2Q_2^2}{B^2Q_1Q_3}\right) \qquad (4.51c)$$

On substituting into (63b) and simplifying, we get $V_{aD}=\frac{Q_5\sqrt{3}}{3Q_3Q_6}$ (4.51d)

The equation for determining the dynamic buckling load λ_D is obtained by multiplying (62a) (evaluated at $\lambda=\lambda_D$) by 3 and getting $3\delta=V_{aD}\left\{3\ \left(e_1+e_2Va_D\right)+3e_3V_{aD}^2\ \right\}$ (4.52a)

If we make $3e_3V_{aD}^2$ the subject in (63a) and substitute same into (4.52a) and simplify, we get

$$3\delta=V_{aD}\,(2e_1+e_2V_{aD})=2V_{aD}\left(1-\frac{C_2V_{aD}}{2C_1^2}\right) \qquad (4.52b)$$

If we substitute for C_1, C_2 and V_{aD} into (4.52b) from (4.49b) and (4.51d) respectively and simplify, we get

$$\lambda_D\in=\frac{2\sqrt{3}\left(\Omega^2-\phi^2\right)}{9\bar{a}\lambda_C\left(\frac{\alpha}{2}+n^2\xi\right)Q_1Q_3Q_6}\left[1-\frac{\sqrt{3}Q_2Q_5\left(\Omega^2-\phi^2\right)}{6\bar{a}\lambda_C\left(\frac{\alpha}{2}+n^2\xi\right)Q_1^2Q_3Q_6}\right] \qquad (4.53)$$

5.0 Analysis of result and conclusion

The result (4.53) is implicit in the buckling load parameter λ_D and is asymptotically valid for small values of $\in$ and $\bar{\mathrm{a}}$ .All along, we have tacitly implied that the denominators of each of the terms occurring in this analysis do not vanish so that the result is uniformly valid and finite for all values of the parameters involved in the formulation. Guided by Koiter's theory [5,6] , the applicability of (4.53) is limited to imperfections whose amplitude is less than one-half of the shell thickness , *i.e.* $\in<\frac{1}{2}$. By setting $\phi=0$, we automatically realize the corresponding step loading result. Because of the inclusion of the term $\sum_{m=1,3,5,\cdots}^{\infty}V_{23}^{(3)}(t,\tau)\sin\left(\frac{m\pi}{2}\right)$, as in (4.46), the final result (4.53) is not strictly that expected of buckling modes in the shape of the classical mode. In order words, we may have 'over estimated' the result. Such a result can however be obtained from (4.53) by either setting $Q_{17}^{(m)}=0$ or setting $\mathrm{S}_{20}=S_{21}=S_{22}=0$. . The novelty of the analysis is that the result (66) can be significantly improved (if desired) by the inclusion of the buckling mode $\sum_{m=1,3,5,\cdots}^{\infty}V_{12}^{(2)}(t,\tau)\sin\left(\frac{m\pi}{2}\right)$, which we earlier on neglected. We have limited our investigation to the case where damping is of the order of the amplitude of the imperfection. This need not be the case always .It would be much more rewarding and perhaps satisfying to investigate the case where the small damping is independent and unrelated to the amplitude of the imperfection .The Mathematical analysis in this case is slightly different . This and other findings will be reported elsewhere.

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