Journal of the Nigerian Association of Mathematical Physics
Volume 10 (November 2006), **315 - 326**

Perturbation analysis on the dynamic buckling of a lightly damped spherical cap modulated by a slowly varying sinusoidal load (1)

A. M. Ette
Department of Mathematics and Computer Science
Federal University of Technology, Owerri, Nigeria
e-mail: tonimonsette@ yahoo.com

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Abstract

This investigation makes a conscious effort at analytical determination of the dynamic buckling load of an imperfect lightly damped spherical cap modulated by a sinusoidally slowly varying dynamic load. Essentially, the formulation is that of an elastic nonlinear oscillatory system, with small perturbations and with coefficients that are harmonically and dynamically slowly varying. The imperfection is discretized into an axisymmetric and a non-axisymmetric mode which are also the shapes of the equally discretized buckling modes. The dynamic buckling load is obtained and is related to the static buckling load. This by-passes the labour of repeating the entire process for different imperfection parameters.

1.0 Introduction

The extent of the dynamic stability of an elastic structure under an external load is an important criterion normally sought for when forces or stresses of excessive magnitude are impacted on any elastic structure. In this investigation, we intend to evaluate the dynamic buckling load of an imperfect spherical cap acted upon by a dynamically slowly varying sinusoidal load applied shortly after the initial time. The inclusion of damping is essential because most physically realistic structures in dynamical systems are never devoid of some element of damping.

The original investigation into the dynamic stability of an imperfect spherical cap under a step load was started by Danielson [1] who used the concept of Mathieu - type of instability in perturbation procedures to determine the dynamic buckling load of the undamped structure. As earlier indicated, the problem we are about to solve has dynamically slowly varying sinusoidal coefficients, which, in the best of times, can always be solved (as in Danielson's case) , using the concept of Mathieu - type of instability . We shall however drop the concept of Mathieu-type of instability for , as noted by Budiansky [2, page 100], Mathieu - type of instability is usually associated with many cycles of oscillation as opposed to just one cycle of oscillation that normally triggers off dynamic buckling .

2.0 Formulation

The relevant differential equations were derived by Danielson [1], when he considered the normal displacement W(x, y, $\tilde{t}$) of a point on the shell surface to be given by

$$W\left(x, y, \tilde{t}\right) = \xi_0\left(\tilde{t}\right) W_0 + \xi_1\left(\tilde{t}\right) W_1 + \xi_2\left(\tilde{t}\right) W_2 \qquad (2.1)$$

where W_0 , W_1 and W_2 are the symmetric pre-buckling mode, an axisymmetric buckling mode and a non - axisymmetric buckling mode respectively while $\xi_0(\tilde{t})$, $\xi_1(\tilde{t})$ and $\xi_2(\tilde{t})$ are the associated time dependent amplitudes of the respective modes . Danielson discretized the imperfection $\overline{W}(x, y)$ into the shapes of the buckling modes, namely

$$\overline{W}(x, y) = \overline{\xi}_1 \overline{W_1} + \overline{\xi}_2 \overline{W}_2 \qquad (2.2)$$

where $\overline{\xi}_1$ and $\overline{\xi}_2$ are two small but Mathematically unrelated parameters signifying the amplitudes of W_1 and W_2 respectively in (2.2) .We shall let $\overline{\xi}_1$ and $\overline{\xi}_2$ satisfy the inequalities $0<\overline{\xi}_1<1$, $0<\overline{\xi}_2<1$. Danielson substituted equations (2.1) and (2.2) into the equation of compatibility and equation of dynamic equilibrium characterising an imperfect spherical cap and obtained the following equations which we have slightly modified by adding a sinusoidally slowly varying load function $\lambda \cos \delta_0 \tilde{t}$ thus:

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{d\tilde{t}^2} + \xi_0 = \lambda \cos \delta_0 \tilde{t} \ , 0 < \delta_0 < 1 , \ \tilde{t} > 0 \qquad (2.3)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{d\tilde{t}^2} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \overline{\xi}_1\xi_1 \ , \tilde{t} > 0 \qquad (2.4)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{d\tilde{t}^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \overline{\xi}_2\xi_2 \ , \tilde{t} > 0 \qquad (2.5)$$

$$\xi_i(0) = \frac{d\xi_i(0)}{d\tilde{t}} = 0 \quad , i = 0,1,2 \qquad (2.6)$$

Here $k_1 > 0$, $k_2>0$ and λ is a nondimensional load amplitude satisfying the inequality $0 < \lambda < 1$ and $\omega_i, i = 0,1,2$ are the circular frequencies of the associated modes. We shall assume $0 < \omega_1 < \omega_0$,$0 < \omega_2 < \omega_0$ .To arrive at equations (2.3) - (2.6), Danielson ignored all nonlinearities greater than the quadratic and solved the equations for step lading case subject to the following assumptions:

(a) Quantities of the order shell thickness divided by the radius can be neglected compared to unity.
(b) Tangential and boundary effects are negligible.
(c) $\overline{\xi}_1$ can be set equal to zero assuming that non-axisymmetric imperfections are the main cause of the reduction in the elastic strength of the structure.
(d) The effects of the quadratic term $k_1\xi_1^2$ may be neglected compared to the effects of coupling between the buckling modes for initial buckling behaviour.
(e) The ratio of the subsequent frequencies namely $\frac{\omega_i}{\omega_{i-1}}$ is taken as $(1-\nu)$ where ν is the Poisson's ratio .

In line with Danielson's assumption (c), we shall set $\overline{\xi}_1 = 0$ but shall admit all nonlinearities in the formulation including the quadratic term $k_1\xi_1^2$ that was ignored by Danielson. In our quest for solution, we are to determine a particular value of λ ,represented as λ_D , called the dynamic buckling load that satisfies the inequality $0 < \lambda_D < \lambda_s < \lambda_c \le 1$ where λ_s and λ_D are the corresponding static buckling load and the classical buckling load respectively .We define λ_D as the maximum load parameter for which the solution of the problem remains bounded .To solve the problem ,we shall first determine uniformly valid asymptotic expressions of the time dependent displacement amplitudes $\xi_0(\tilde{t})$, $\xi_1(\tilde{t})$ and $\xi_2(\tilde{t})$ .We shall next determine the maxima of these amplitudes and also determine the net maximum amplitude which is the sum of the maxima of

$\xi_1(\tilde{t})$ and $\xi_2(\tilde{t})$. We shall lastly determine the dynamic buckling load via a suitable maximization procedure. For simplicity of analysis, the pre - buckling amplitude $\xi_0(\tilde{t})$, is here considered not damped while only the buckling amplitudes $\xi_1(\tilde{t})$ and $\xi_2(\tilde{t})$ are considered lightly and viscously damped. The damping is taken proportional to the first degree of the velocity of each amplitude.

3.0 Solution of the problem

We shall let $\bar{t} = \omega_0\tilde{t}$ so that $\cos\delta_0\tilde{t} = \cos\delta\,\bar{t}$, $\delta = \dfrac{\delta_0}{\omega_0}, 0 < \delta < 1$. Thus we have

$$\frac{d(\;)}{d\tilde{t}} = \omega_0\frac{d(\;)}{d\bar{t}}\;;\;\frac{d^2(\;)}{d\tilde{t}^2} = \omega_0^2\frac{d^2(\;)}{d\bar{t}^2} \tag{3.1}$$

On substituting (3.1) into (2.3), we get

$$\frac{d^2\xi_0}{d\bar{t}^2} + \xi_0 = \cos\delta\bar{t};\;\; \xi_0(0) = \frac{d\xi_0(0)}{d\bar{t}} = 0 \tag{3.2a}$$

The solution of (3.2a) is

$$\xi_0(\bar{t}) = \frac{\cos\delta\,\bar{t} - \cos\bar{t}}{1-\delta^2} \tag{2.2b}$$

We next substitute (3.1) and (3.2b) into (2.1)-(2.6), set $\bar{\xi}_1 = 0$ and add the viscous damping terms $2\delta\dfrac{d\xi_1}{d\bar{t}}$ *and* $2\delta\dfrac{d\xi_2}{d\bar{t}}$ to the simplified equations corresponding to (2.4) and (2.5) respectively and get the following

$$\frac{d^2\xi_1}{d\bar{t}^2} + 2\delta\frac{d\xi_1}{d\bar{t}} + Q^2\xi_1\left[1 - \frac{T\in(1+\delta^2+\cdots)(\cos\delta\,\bar{t} - \cos\bar{t})}{Q^2}\right] - k_1Q^2\xi_1^2 + k_2Q^2\xi_2^2 + = 0 \tag{3.3a}$$

$$\frac{d^2\xi_2}{d\bar{t}^2} + 2\delta\frac{d\xi_2}{d\bar{t}} + R^2\xi_2\left[1 - \frac{\in(1+\delta^2+\cdots)(\cos\delta\,\bar{t} - \cos\bar{t})}{R^2}\right] + R^2\xi_1\xi_2 \tag{3.b}$$

$$= \in\bar{\xi}_2(1+\delta^2+\cdots)(\cos\delta\,\bar{t} - \cos\bar{t})$$

$$_{\mathrm{i}}(0) = \frac{d\xi_{\mathrm{i}}(0)}{d\bar{t}} = 0 \;\; \mathrm{i} = 1,2.; \tag{3.3c}$$

$$Q = \left(\frac{\omega_1}{\omega_0}\right), \mathrm{R} = \left(\frac{\omega_2}{\omega_0}\right), \in = \left(\frac{\omega_2}{\omega_0}\right)^2\lambda, \mathrm{T} = \left(\frac{\omega_1}{\omega_2}\right)^2, 0 < \in << 1 \tag{3.3d}$$

We emphasize that $\in$ and δ are two small but mathematically unrelated parameters such that with respect to them, the analysis becomes a two—small -parameter problem with sinusoidally and dynamically slowly varying coefficients. The case where $\in$ and δ are mathematically related is equally interesting and shall be reported elsewhere. The analysis here is in line with those of Aksogan and Sofiyev [3], Wang and Tian [4-6] and Karagiozova and Jones [7]. We shall now adopt the following time scales

$$\tau = \delta\,\bar{t}, \mathrm{t} = \bar{t} + \left(\frac{\Omega_1(\tau)\in + \Omega_2(\tau)\in^2 + \cdots}{\delta}\right), \Omega_{\mathrm{i}}(0) = 0, i = 1,2,3,\cdots \tag{3.4}$$

Thus we have

$$\frac{d\xi_\alpha}{d\bar{t}} = (1 + \Omega_1'\in + \Omega_2'\in^2 + \cdots)\xi_{\alpha,\mathrm{t}} + \delta\xi_{\alpha,\tau}\;\;, \alpha = 1,2 \tag{3.5a}$$

$$\frac{d\xi^2{}_\alpha}{d\bar{t}^2} = \left(1+\Omega_1' \in +\Omega_2' \in^2 +\cdots\right)^2 \xi_{\alpha,tt} + 2\left(1+\Omega_1' \in +\Omega_2' \in^2 +\cdots\right)\xi_{\alpha,t\tau} + \delta^2 \xi_{\alpha,\tau\tau} \qquad (3.5b)$$
$$+\delta\left(\Omega_1'' \in +\Omega_2'' \in^2 +\cdots\right)\xi_{\alpha,t}$$

Here , a subscript following a comma indicates partial differentiation with respect to that subscript while $\frac{d(\;)}{d\tau} \equiv (\;)'$ .We now let

$$\xi_1(\bar{t}) = \xi_1(t,\tau) = \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} A^{ij}(t,\tau) \in^i \delta^j, \xi_2(\bar{t}) = \xi_2(t,\tau) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} G^{ij}(t,\tau) \in^i \delta^j \qquad (3.6)$$

where the ij on A^{ij} and B^{ij} are superscripts and not powers. By substituting (3.5a,b) into (3.3a-d) , using (3.6),and equating equations of powers of $\in \delta$ we have the following sequence of equations :

$$NA^{2o} \equiv A^{20}_{,tt} + Q^2 A^{20} = -k_2 Q^2 \left(G^{10}\right)^2 \qquad (3.7)$$

$$NA^{21} = -2k_2 G^{10} G^{11} - 2A^{20}_{,t} - 2A^{20}_{,t\tau} \qquad (3.8)$$

$$NA^{22} = -k_2\left[2G^{10}G^{12} + \left(G^{11}\right)^2\right] - 2A^{21}_{,t} - 2A^{21}_{,t\tau} - A^{20}_{,\tau} \qquad (3.9)$$

$$MG^{10} \equiv G^{10}_{,tt} + R^2 G^{10} = \bar{\xi}_2\left(\cos\tau - \cos\bar{t}\right) \qquad (3.10)$$

$$MG^{11} = -2G^{10}_{,t} - 2G^{10}_{,t\tau} \qquad (3.11)$$

$$MG^{12} = -2G^{11}_{,t} - 2G^{11}_{,t\tau} - G^{10}_{,\tau} + \bar{\xi}_2\left(\cos\tau - \cos\bar{t}\right) \qquad (3.12)$$

$$MG^{20} = \left(\cos\tau - \cos\bar{t}\right)G^{10} - 2\Omega_1' G^{10}_{,tt} \qquad (3.13)$$

$$MG^{21} = \left(\cos\tau - \cos\bar{t}\right)G^{11} - 2G^{20}_{,t} - 2\Omega_1' G^{10}_{,t} - 2\Omega_1' G^{11}_{,t} - 2\Omega_1' G^{10}_{,t\tau} - \Omega_1'' G^{10}_{,t} - 2G^{20}_{,t\tau} \qquad (3.14)$$

$$MG^{22} = \left(\cos\tau - \cos\bar{t}\right)G^{12} - 2G^{21}_{,t} - 2\Omega_1' G^{11}_{,t} - 2\Omega_1' G^{12}_{,t} - 2\Omega_1' G^{11}_{,t\tau} - \Omega_1'' G^{11}_{,t} - 2G^{21}_{,t\tau} - 2G^{20}_{,\tau} \qquad (3.15)$$

The initial conditions, which are evaluated at $(t,\tau) = (0,0)$ are as follows:

$$A^{2j} = 0,\; j = 0,1,2,\cdots \;;\; A^{20}_{,t} = 0 \qquad (3.16a)$$

$$A^{2r}_{,t} + \Omega_1'(0) A^{1r}_{,t} + A^{2s}_{,\tau} = 0, \mathrm{s = r-1, r = 1,2,3,\cdots} \qquad (3.16b)$$

$$G^{ij} = 0, G^{10}_{,t} = 0 \;;\; G^{1r}_{,t} + \Omega_1'(0) G^{1s}_{,\tau} = 0, \mathrm{s = r-1, r = 1,2,3,\cdots} \qquad (3.16c)$$

$$G^{2r}_{,t} + \Omega_1'(0) G^{1r}_{,t} + G^{2s}_{,\tau} = 0, \mathrm{s = r-1, r = 1,2,3,\cdots} \qquad (3.16d)$$

From the second of (3.4), we know that

$$\cos t = \cos\left\{\bar{t} + \left(\frac{\Omega_1 \in + \Omega_2 \in^2 + \cdots}{\delta}\right)\right\} \qquad (3.17a)$$

Thus ,if $\in$ is small enough compared to unity , we can conveniently take the approximation

$$\cos\bar{t} \cong \cos t \qquad (3.17b)$$

Now substituting (3.17b) into (3.10),and solving, using (3.16c) for i = 1, j = 0,we get

$$G^{10} = a_{10}(\tau)\cos Rt + b_{10}(\tau)\sin Rt + \bar{\xi}\left(\frac{\cos\tau}{R^2} - \frac{\cos t}{R^2 - 1}\right) \qquad (3.18a)$$

$$a_{10}(0) = r_1\bar{\xi},\; r_1 = \frac{1}{R^2-1} - \frac{1}{R^2}, b_1(0) = 0 \qquad (3.18b)$$

where we have , without loss of generality, taken $\bar{\xi}_2 \equiv \bar{\xi}$ .We now substitute for terms into (3.11) and to ensure a uniformly valid solution in terms of the time scale t , equate to zero the coefficients of cosRt and sinRt and get respectively $b'_{10} + b_{10} = 0$ and $a'_{10} + a_{10} = 0$ (3.19c)

The solutions of (3.18c) subject to (3.18b) are

$$b_{10}(\tau) \equiv 0 \; ; \; a_{10}(\tau) = a_{10}(0)e^{-\tau} \, , b'_{10}(0) = 0, a'_{10}(0) = -r_1\bar{\xi} \tag{3.19d}$$

Thus we have
$$G^{10} = a_{10}(\tau)\cos Rt + \bar{\xi}\left(\frac{\cos\tau}{R^2} - \frac{\cos t}{R^2 - 1}\right) \tag{3.19e}$$

The solution of the remaining equation in (3.11) is

$$G^{11}(t,\tau) = a_{11}(\tau)\cos Rt + b_{11}(\tau)\sin Rt \, ; \; a_{11}(0) = 0 \; ; b_{11}(0) = -\frac{r_1\bar{\xi}}{R} \tag{3.20}$$

On substituting for terms into (3.12) and equating to zero the coefficients of cosRt and sinRt, we get respectively

$$b'_{11} + b_{11} = 0 \quad \text{and} \quad a'_{11} + a_{11} = 0 \tag{3.21a}$$

The solutions of (3.21a) subject to (3.20) are

$$b_{11}(\tau) = e^{-\tau}\int_0^\tau \left[b_{11}(0) - \frac{a'(s)e^{s}ds}{R} \right] \; ; \; a_{11}(\tau) \equiv 0 \tag{3.21b}$$

Thus we have
$$G^{11}(\tau) = b_{11}(\tau)\sin Rt \tag{3.21c}$$

We next substitute for G^{10} into (3.7) and simplify to get

$$NA^{20} = -k_2Q^2\left[L_1 + \frac{a_{10}^2\cos 2Rt}{2} + \frac{\bar{\xi}^2\cos 2\tau}{2R^4} + 2a_{10}\bar{\xi}\cos\tau\cos Rt \right. \tag{3.22a}$$

$$\left. - \frac{a_{10}\bar{\xi}}{R^2}\{\cos(1+R)t + \cos(1-R)t\} - \frac{2\bar{\xi}^2\cos\tau\cos t}{R^2(R^2-1)} + \frac{\bar{\xi}^2\cos 2t}{2(R^2-1)^2} \right], \quad R \neq 1 \tag{3.22c}$$

where

$$L_1(\tau) = \frac{a_{10}^2}{2} + \bar{\xi}^2\left\{\frac{1}{2R^4} + \frac{1}{2(R-1)^2}\right\}, \; L_1(0) = \bar{\xi}^2 l_1$$

$$l_1 = \left[\frac{1}{2}\left\{\frac{1}{(R-1)^2} - \frac{1}{R^2}\right\}^2 + \frac{1}{2R^4} + \frac{1}{2(R^2-1)^2} \right] ; L'_1(0) = -\left(\frac{r_1\bar{\xi}}{R}\right)^2 \tag{3.22b}$$

We now solve (3.22a-c) subject to (3.16a) (for j = 0) and get

$$A^{20}(t,\tau) = \gamma_1(\tau)\cos Qt + \beta_1(\tau)\sin Qt - k_2Q^2\left[\frac{L_1}{Q^2} + \frac{a_{10}^2\cos 2Rt}{2(Q^2 - 4R^2)} + \frac{\bar{\xi}^2\cos 2\tau}{2(QR^2)^2} \right.$$

$$+ \frac{2\bar{\xi}a_{10}\cos\tau\cos Rt}{Q^2 - R^2} - \frac{a_{10}\bar{\xi}}{R^2}\left\{\frac{\cos(1+R)t}{Q^2 - (1+R)^2} + \frac{\cos(1-R)t}{Q^2 - (1-R)^2}\right\} - \frac{2\bar{\xi}^2\cos\tau\cos t}{R^2(R^2-1)(Q^2-1)} \tag{3.23a}$$

$$\left. + \frac{\bar{\xi}^2\cos 2t}{2(R^2-1)^2(Q^2-4)} \right]$$

$$\gamma_1(0)=\bar{\xi}^2 k_2 Q^2 l_2\ ;\ l_2=\left[\frac{l_1}{Q^2}+\frac{r_1^2}{2(Q^2-4R^2)}+\frac{1}{2Q^2R^4}-\frac{2}{R^2(R^2-1)(Q^2-1)}\right.$$

$$\left.+\frac{1}{2(R^2-1)^2(Q^2-4)}+\frac{2r_1}{Q^2-R^2}-\frac{r_1}{R^2}\left\{\frac{1}{Q^2-(1+R)^2}+\frac{1}{Q^2-(1-R)^2}\right\}\right],\ \beta_1(0). \tag{3.23b}$$

where Q ≠ 2, Q ≠ R, 2R ≠ Q, Q ≠ 1,Q ≠ (1+ R) and Q ≠ (1-R) .

We shall next substitute into (3.8) and to ensure a uniformly valid solution, equate to zero the coefficients of cosQT and sinRt and get respectively

$$\beta_1'+\beta_1=0 \text{ and } \gamma_1'+\gamma_1=0 \tag{3.24a}$$

The solutions of (3.24a) are

$$\beta_1(\tau)\equiv 0,\ \gamma_1(\tau)=\gamma_1(0)e^{-\tau},\ \gamma_1'(0)=-\gamma_1(0)=-\bar{\xi}^2 k_2 Q^2 l_2 \tag{3.24b}$$

The remaining equation in (3.8) is now solved to get

$$A^{21}(t,\tau)=\gamma_2(\tau)\cos Qt+\beta_2(\tau)\sin Qt-2k_2Q^2\left[\frac{a_{10}b_{11}}{2R}\left\{\frac{\sin(R-Q)t}{2Q-R}-\frac{\sin(R+Q)t}{2Q+R}\right\}\right.$$

$$\left.+\bar{\xi}b_{11}\left\{\left\{\frac{\sin Rt\cos\tau}{R^2(Q^2-R^2)}-\frac{1}{2(R^2-1)}\left\{\frac{\sin(1+R)t}{Q^2-(1+R)^2}+\frac{\sin(R-1)t}{Q^2-(R-1)^2}\right\}\right\}\right\}\right]$$

$$+2k_2Q^2\left[-\frac{R(a_{10}^2+a_{10}a_{10}')\sin 2Rt}{(Q^2-4R^2)^2}+\frac{2\bar{\xi}^2(\cos\tau-\sin\tau)\sin t}{R^2(R^2-1)(Q^2-1)^2}\right.$$

$$+\frac{2R\bar{\xi}}{Q^2-R^2}\left\{(a_{10}\cos\tau)'-a_{10}\cos\tau\right\}\sin Rt$$

$$\left.-\frac{2\bar{\xi}\sin 2t}{2(R^2-1)(Q^2-4)^2}+\frac{\bar{\xi}}{R^2}\left\{\frac{(1+R)\sin(1+R)t}{Q^2-(1+R)^2}+\frac{(1-R)\sin(1-R)t}{Q^2-(1-R)^2}\right\}(a_{10}'+a_{10})\right],\ Q\neq\frac{R}{2},\ Q\neq 2 \tag{3.25a}$$

$$\gamma_2(0)=0;\ \beta_2(0)=2k_2Q\bar{\xi}^2 l_4,\ l_4=\left[\frac{1}{2}\left(\frac{r_1}{R}\right)^2\left\{\frac{R-Q}{2Q-R}-\left(\frac{Q+R}{2Q+R}\right)\right\}-\frac{r_1}{R}\left\{\left\{\frac{1}{R(Q^2-R^2)}\right.\right.\right.$$

$$\left.\left.-\frac{1}{2(R^2-1)}\left\{\frac{(1+R)}{Q^2-(1+R)^2}+\frac{(R-1)}{Q^2-(R-1)^2}\right\}\right\}\right\}-\left\{\frac{2}{R^2(R^2-1)(Q^2-1)^2}-\frac{2}{(R^2-1)(Q^2-1)^2}\right.$$

$$\left.\left.-\frac{2Rr_1}{Q^2-R^2}+\frac{l_2}{2}\right\}-\frac{1}{2}\left\{\left\{\frac{r_1^2}{(QR)^2}+\frac{r_1}{Q^2-4R^2}+\frac{2r_1}{(Q^2-R^2)}+\frac{r_1}{R^2}\left\{\frac{1}{Q^2-(1+R)^2}+\frac{1}{Q^2-(1-R)^2}\right\}\right\}\right\}\right] \tag{3.25b}$$

On substituting into (3.13) for G^{10} from (3.18e) and equating to zero the coefficient of cos*R*t, we have

$$\Omega'(\tau)=-\frac{a_{10}\cos\tau}{2Q},\ \Omega'(0)=-\frac{\bar{\xi}r_1}{2Q^2},\ \ \Omega''(0)=\frac{\bar{\xi}r_1}{2Q^2} \tag{3.26a}$$

The remaining equation in the substitution into (3.13) is

$$MG^{20}=\bar{\xi}\left[l_5+\frac{\cos 2\tau}{2R^2}-\frac{\cos 2t}{2(R^2-1)}+\left(l_6\cos\tau-\frac{2\Omega'}{(R^2-1)}\right)\cos t\right.$$

$$\left.+\frac{a_{10}}{2\bar{\xi}}\{\cos(Q+1)t+\cos(Q-1)t\}\right] \tag{3.26b}$$

$$l_5=\frac{1}{2R^2}-\frac{1}{2(R^2-1)}\ ,\ \ l_6=\frac{1}{R^2}-\frac{1}{(R^2-1)} \tag{3.26c}$$

The solution of (3.26a-c), subject to (3.16c) for *i* = 2, j = 0 and (3.16e) for *r* = 1, is

$$G^{20}=a_{20}(\tau)\cos Rt+b_{20}(\tau)\sin Rt+\bar{\xi}\left[\frac{l_5}{R^2}+\frac{\cos 2\tau}{2R^4}-\frac{\cos 2t}{2(R^2-1)(R^2-4)}\right.$$
$$\left.+\left\{l_6\cos\tau-\frac{2\Omega'}{(R^2-1)}\right\}\frac{\cos t}{(R^2-1)}+\frac{a_{10}}{2\bar{\xi}}\left\{\frac{\cos(1+Q)t}{R^2-(1+Q)^2}+\frac{\cos(Q-1)t}{R^2-(Q-1)^2}\right\}\right] \quad (3.26d)$$

$$a_{20}(0)=\bar{\xi}l_7+\bar{\xi}^2 l_8\ ,\ \mathrm{l}_7=-\left[\frac{l_5}{R^2}+\frac{1}{2R^4}-\frac{1}{2(R^2-1)(R^2-4)}+\frac{l_6}{R^2-1}\right.$$
$$\left.+\frac{r_1}{2}\left\{\frac{1}{R^2-(1+Q)^2}+\frac{1}{R^2-(Q-1)^2}\right\}\right],\ \mathrm{l}_8=\frac{r_1}{Q^2(R^2-1)^2};\ \mathrm{b}_{20}(0)=0 \quad (3.26e)$$

where $\mathrm{R}\neq 2,\ \mathrm{R}\neq(1+\mathrm{Q})\ ,\mathrm{R}\neq(\mathrm{Q}\text{-}1)$

We shall next substitute for terms into (3.24) and to ensure a uniformly valid solution in terms of the time scale t , equate to zero the coefficients of cosRt and sinRt to get respectively

$$b'_{20}+b_{20}=0\ \text{ and }\ a'_{20}+a_{20}=H(\tau) \quad (3.27a)$$

$$H(\tau)=-\frac{1}{2R}\left[2\Omega' R(a_{10}+a'_{10})+Ra_{10}\Omega''+b_{11}\cos\tau+2\Omega' R^2 b_{11}\right] \quad (3.27b)$$

The solutions of (3.27a,b) subject to (3.26e) are

$$b_{20}(\tau)=0\ ,\ \mathrm{a}_{20}(\tau)=e^{-\tau}\int_0^{\tau}\left[a_{20}(0)+H(s)e^{s}ds\right] \quad (3.27c)$$

The remaining equation in the substitution into (3.14) is

$$MG^{21}=\bar{\xi}L_3\sin t-\frac{2\bar{\xi}\sin 2t}{(R^2-1)(R^2-4)}-\frac{b_{11}}{2}\{\sin(1+R)t+\sin(R-1)t\}$$
$$+L_4\sin(Q+1)t+L_5\sin(Q-1)t \quad (3.28a)$$

where

$$L_3=\left[2\left(l_6\cos\tau-\frac{2\Omega'}{R^2-1}\right)\left(\frac{1}{R^2-1}\right)-\frac{2\Omega'}{R^2-1}-2\left\{l_6\sin\tau+\frac{2\Omega''}{R^2-1}\right\}\left(\frac{1}{R^2-1}\right)-\frac{\Omega''}{R^2-1}\right] \quad (3.28)$$

$$L_4=-\frac{1}{2}\left\{\frac{a_{10}(Q+1)}{R^2-(Q+1)^2}+\frac{a'_{10}(Q+1)}{R^2-(Q+1)^2}\right\},\ \mathrm{L}_5=-\frac{1}{2}\left\{\frac{(Q-1)(a_{10}+a'_{10})}{R^2-(Q-1)^2}\right\} \quad (3.28c)$$

$$L_3(0)=l_{11}+\bar{\xi}l_{12},\ \mathrm{l}_{11}=\frac{2l_6}{R^2-1},\ \mathrm{l}_{12}=\frac{r_1}{2Q^2}\left(\frac{4}{R^2-1}+\frac{1}{2}\right)+\frac{r_1}{2Q^2}\left(1+\frac{4}{R}\right) \quad (3.28d)$$

$$L_4(0)=L_5(0)=0 \quad (3.28e)$$

The solution of (3.28a-e) is

$$G^{21}(t,\tau)=a_{21}(\tau)\cos Rt+b_{21}(\tau)\sin Rt+\frac{\bar{\xi}L_3\sin t}{R^2-1}-\frac{2\bar{\xi}\sin 2t}{(R^2-1)(R^2-4)^2}$$
$$-\frac{b_{11}}{2}\left\{\frac{\sin(R-1)t}{2R-1}-\frac{\sin(R+1)t}{2R+1}\right\}+\frac{L_4\sin(Q+1)t}{R^2-(Q+1)^2}++\frac{L_5\sin(Q-1)t}{R^2-(Q-1)^2} \quad (3.29a)$$

$$a_{21}(0)=0\ ,\ \mathrm{b}_{21}(0)=-\frac{1}{R}\left[\frac{\bar{\xi}L_3}{R^2-1}-\frac{4\bar{\xi}}{(R^2-1)(R^2-4)^2}-\frac{b_{11}}{2}\left\{\frac{R-1}{2R-1}-\frac{R+1}{2R+1}\right\}\right.$$

$$-a'_{20}(0)-\frac{2\bar{\xi}\Omega''_1}{R^2-1}-\frac{a'_{10}}{2}\left\{\frac{1}{\mathrm{R}^2-(Q+1)^2}+\frac{1}{\mathrm{R}^2-(Q-1)^2}\right\}\Bigg]\Bigg]\Bigg|_{\tau=0} \quad (3.29b)$$

We can determine $a_{21}(\tau)$ and $\mathrm{b}_{21}(\tau)$ in full by substituting into (3.15) and demanding that a uniformly valid solution be obtained. However their full determination is not necessary. Thus we write

$$\xi_1(t,\tau)=\in^2\left(A^{20}+A^{21}\delta+\cdots\right)+\cdots;\xi_2(t,\tau)=\in\left(G^{10}+\delta G^{11}\right)+\in^2\left(G^{20}+\delta G^{21}\right)+\cdots \quad (3.30)$$

3.0 Maximum displacement and dynamic buckling load

The condition for the attainment of maximum amplitudes of the buckling modes is

$$\left(1+\Omega'_1\in+\Omega'_2\in^2+\cdots\right)\xi_{j,t}+\delta\xi_{j,\tau}=0,\ \mathrm{j}=1,2 \quad (4.1)$$

We shall let $t_a, \bar{t}_a$ and τ_a be the critical values of t, $\bar{t}$ and τ respectively for $\xi_1(t,\tau)$ at maximum displacement while $t_c, \bar{t}_c$ and τ_c are the corresponding values of t, $\bar{t}$ and τ respectively for $\xi_2(t,\tau)$. We assume the following asymptotic series:

$$t_a=t_0^{(1)}+\delta t_{01}^{(1)}+\in\left(t_{10}^{(1)}+\delta t_{11}^{(1)}\right)+\cdots;\bar{t}_a=\bar{t}_0^{(1)}+\delta\bar{t}_{01}^{(1)}+\in\left(\bar{t}_{10}^{(1)}+\delta\bar{t}_{11}^{(1)}\right)+\cdots \quad (4.2a)$$

$$\tau_a=\delta\bar{\mathrm{t}}_{\mathrm{a}}=\delta\left[\bar{t}_0^{(1)}+\delta\bar{t}_{01}^{(1)}+\in\left(\bar{t}_{10}^{(1)}+\delta\bar{t}_{11}^{(1)}\right)+\cdots\right];\ \mathrm{t_c}=t_0^{(2)}+\delta t_{01}^{(2)}+\in\left(t_{10}^{(2)}+\delta t_{11}^{(2)}\right)+\cdots \quad (4.2b)$$

$$\bar{t}_c=\bar{t}_0^{(2)}+\delta\bar{t}_{01}^{(2)}+\in\left(\bar{t}_{10}^{(2)}+\delta\bar{t}_{11}^{(2)}\right)+\cdots\ ;\ \tau_{\mathrm{c}}=\delta\bar{\mathrm{t}}_{\mathrm{c}}=\delta\left[\bar{t}_0^{(2)}+\delta\bar{t}_{01}^{(2)}+\in\left(\bar{t}_{10}^{(2)}+\delta\bar{t}_{11}^{(2)}\right)+\cdots\right] \quad (4.2c)$$

To determine some of the terms in (4.2a), we determine (4.2b) , for *j* = 1, at the critical point. Here, we equate the coefficient of $\in$ and get $A_{,t}^{20}\left(t_0^{(1)},0\right)=0$. This gives the equation for determining $t_0^{(1)}$ as

$$\gamma_1(0)\sin t_0^{(1)}-k_2Q\left[\frac{2\bar{\xi}^2\sin t}{R^2(R^2-1)(Q^2-1)}-\frac{Ra_{10}^2\sin 2Rt}{Q^2-4R^2}+\frac{a_{10}\bar{\xi}}{R^2}\left\{\frac{(R+1)\sin(R+1)t}{Q^2-(R+1)^2}\right.\right.$$
$$\left.\left.+\frac{(1-R)\sin(1-R)t}{Q^2-(1-R)^2}\right\}-\frac{2Ra_{10}\bar{\xi}\sin Rt}{Q^2-R^2}-\frac{\bar{\xi}^2\sin 2t}{(R^2-1)^2(Q^2-4)}\right]\Bigg|_{t=t_0^{(1)},\,\tau=0}=0 \quad (4.3a)$$

For an approximate value of $t_0^{(1)}$, we can maintain just the first terms in theTaylor series expansion of the terms in (4.aa) and get

$$t_0^{(1)}\cong\sqrt{\frac{r_2}{r_3}} \quad (4.3b)$$

In the determination of the maximum value ξ_{1a} of $\xi_1(t.\tau)$, we shall need the following evaluations:

where

$$r_2=\left[Q^2l_2+\frac{2R^2r_1^2}{Q^2-4R^2}-\frac{2}{R^2(R^2-1)(Q^2-1)}+\frac{2}{(R^2-1)^2(Q^2-4)}+\frac{2R^2r_1}{Q^2-R^2}\right.$$
$$\left.-\frac{r_1(1+R)^2}{R^2\left\{Q^2-(1+R)^2\right\}}-\frac{(1-R)^2}{R^2\left\{Q^2-(1-R)^2\right\}}\right] \quad (4.3c)$$

$$r_3=\left[\frac{l_2Q^4}{6}+\frac{4R^2}{3(Q^2-4R^2)}+\frac{4}{3(R^2-1)^2(Q^2-4)}+\frac{R^4r_1}{3(Q^2-R^2)^2}\right.$$
$$\left.-\frac{1}{3R^2(R^2-1)(Q^2-1)}-\frac{r_1(1+R)^4}{6R^2\left\{Q^2-(1+R)^2\right\}}-\frac{r_1(1-R)^4}{6R^2\left\{Q^2-(1-R)^2\right\}}\right] \quad (4.3d)$$

$$A_{,\tau}^{20}\left(t_o^{(1)},0\right)=k_2Q^2\bar{\xi}^2S_1,\ \mathrm{S}_1=\left[\frac{r_1^2}{(QR)^2}-l_2\cos Qt+\frac{r_1^2\cos 2Rt}{R^2-(Q-4R)^2}\right.$$

$$\left.\left.-\frac{r_1}{R^2}\left\{\frac{\cos(1+R)t}{Q^2-(+R)^2}+\frac{\cos(1-R)t}{Q^2-(1-R)^2}\right\}\right]\right|_{t=t_0^{(1)}} \qquad (4.3e)$$

$$A^{21}\left(t_0^{(1)},0\right)=2k_2Q^2S_2\ ;\ \mathrm{S}_2=\left[\frac{l_4\sin Qt}{Q}-\left\{\left\{\frac{r_1^2}{2R}\left\{\frac{\sin(Q+R)t}{2Q+R}-\frac{\sin(R-Q)t}{2Q-R}\right\}\right.\right.\right.$$

$$\left.\left.\frac{r_1}{2R}\left\{\frac{\sin Rt}{R^2\left(Q^2-R^2\right)}-\frac{1}{2\left(R^2-1\right)}\left(\frac{\sin(1+R)t}{Q^2-(1+R)^2}+\frac{\sin(R-1)t}{Q^2-(R-1)^2}\right)\right\}\right\}\right\}-\left\{\frac{2\sin t}{R^2\left(R^2-1\right)\left(Q^2-1\right)^2}\right.$$

$$\left.\left.+\frac{4Rr_1\sin t}{Q^2-R^2}+\frac{2\sin 2t}{R\left(R^2-1\right)\left(Q^2-4\right)}\right\}\right]\right|_{t=t_0^{(1)}} \qquad (.4.4)$$

Similarly we have

$$A^{20}\left(t_0^{(1)},0\right)=k_2Q^2\bar{\xi}^2S_3\ ,\ \mathrm{S}_3=\left[\frac{l_1(\cos Qt-1)}{Q^2}+\frac{\cos Qt-\cos t}{2\left(QR^2\right)^2}+\frac{2r_1(\cos Qt-\cos Rt)}{Q^2-R^2}\right.$$

$$+\frac{2(\cos t-\cos Qt)}{R^2\left(R^2-1\right)\left(Q^2-1\right)}+\frac{\cos Qt-\cos 2t}{2\left(R^2-1\right)\left(Q^2-4\right)}+\frac{r_1(\cos Qt-\cos 2Rt)}{Q^2-4R^2}+\frac{r_1}{R^2}\left\{\frac{\cos(1+R)t-\cos Qt}{Q^2-(1+R)^2}\right.$$

$$\left.\left.\left.\frac{\cos(1-R)t-\cos Qt}{Q^2-(1-R)^2}\right\}\right]\right|_{t=t_0^{(1}} \qquad (4.45)$$

The maximum ξ_{1a} of $\xi_1(t.\tau)$ is easily obtained using the first of (3.30) and (4.2a,b) to be

$$\xi_{1a}=\xi_1(t_a,\tau_a)=\in^2\left[A^{20}+\delta\left(\bar{t}_0^{(1)}A_{,\tau}^{20}+A^{21}\right)\right]+\cdots \qquad (4.6a)$$

where (4.6a) is evaluated at $t=t_0^{(1)},\tau=0$. It easily follows from the second of (3.4) evaluated at the critical point of $\xi_1(t,\tau)$ that

$$t_0^{(1)}=\bar{t}_0^{(1)} \qquad (6.6b)$$

Thus we have

$$\xi_{1a}=k_2Q^2\in^2\bar{\xi}^2\left[S_3+\delta\left(\bar{t}_0^{(1)}S_1+2S_2\right)\right]+\cdots \qquad (4.6c)$$

To determine the maximum ξ_{2a} of $\xi_2(t,\tau)$, we substitute for terms into (4.1) for j =2 and use the second of (4.2b) as well as (4.27c) and equate the coefficients of $\in$, $\in\delta$ and $\in^2$ and get respectively

$$G_{,}^{10}=0,\ \mathrm{t}_{01}^{(2)}G_{,\mathrm{tt}}^{10}+G_{,t}^{11}=0\ \text{ and }\ \mathrm{t}_{10}^{(2)}G_{,tt}^{10}+G_{,t}^{20}=0 \qquad (4.7a)$$

We have

$$t_0^{(2)}=\frac{\pi}{R},\ \mathrm{t}_{01}^{(2)}=\frac{r_1\bar{\xi}}{R^2},\ \mathrm{t}_{10}^{(2)}=0 \qquad (4.7b)$$

Thus the maximum ξ_{2a} is now evaluated to give

$$\xi_{2a}=\in\left[G^{10}+\delta\left(\bar{t}_0^{(2)}G_{,\tau}^{10}+G^{11}\right)\right]+\in^2\left[G^{20}+\delta(\bar{t}_{10}^{(2)}G_{,t}^{10}+\bar{t}_0^{(2)}G_{,\tau}^{20}+G^{21})\right]+\cdots \qquad (4.8)$$

where we have written down only the non-vanishing terms in (4.8). To evaluate

$\bar{t}_0^{(2)}$ and $\bar{t}_{10}^{(2)}$ as in (4.8), we note from the second of (3.4) evaluated at the critical point for $\xi_2(t,\tau)$ that

$$\bar{t}_0^{(2)} = t_0^{(2)} = \frac{\pi}{R}, \quad t_{10}^{(2)} = \bar{t}_{10}^{(2)} + \Omega_1'(0)\bar{t}_0^{(2)} \qquad (4.9a)$$

Therefore we have
$$\bar{t}_{10}^{(2)} = -\Omega_1'(0)\bar{t}_0^{(2)} = \frac{r_1\bar{\xi}\pi}{2Q^2R} \qquad (4.9b)$$

The following evaluated terms are as in (4.8):

$$G^{20}\left(t_0^{(2)},0\right) = \bar{\xi}l_{13} + \bar{\xi}l_{14} \qquad (4.10a)$$

$$_{13} = \left[\frac{2l_5}{R^2} + \frac{1}{R^4} - \frac{(1+\cos 2t)}{2(R^2-1)(R^2-4)} + \frac{l_6(1+\cos t)}{(R^2-1)} + \frac{r_1}{2}\left\{\frac{1+\cos(Q+1)t}{R^2-(Q+1)^2}\right.\right.$$
$$\left.\left.\left. + \frac{1+\cos(Q-1)t}{R^2-(Q-1)^2}\right\}\right]\right|_{t=t_0^{(2)}}, l_{14} = \frac{r_1\left(1+\cos t_0^{(2)}\right)}{Q^2\left(R^2-1\right)^2} \qquad (4.10b)$$

$$G_{,\tau}^{(20)}\left(t_0^{(2)},0\right) = \bar{\xi}l_{15} + \bar{\xi}^2 l_{16}, \; l_{15} = -l_7 + \left.\frac{r_1}{2}\left\{\frac{\cos(Q+1)t}{R^2-(Q+)^2} + \frac{\cos(Q-1)t}{R^2-(Q-)^2}\right\}\right|_{t=t_0^{(2)}} \qquad (4.11a)$$

$$l_{16} = -\frac{r_1\cos t_0^{(2)}}{Q^2\left(R^2-1\right)^2} \qquad (4.11b)$$

$$G^{21}\left(t_0^{(2)},0\right) = \bar{\xi}l_{17} \quad , l_{17} = \left[\frac{l_{11}}{R^2-1} - \frac{2\sin 2t_0^{(2)}}{(R^2-1)(R^2 4)} + \frac{l_{12}}{R^2-1} + \frac{r_1}{2R}\left\{\frac{l_{12}}{(R^2-1)}\right.\right.$$
$$\left.\left. + \frac{r_1}{2R}\left(\frac{1}{2R-1} - \frac{1}{2R+1}\right)\sin t_0^{(2)}\right\}\right] \qquad (4.12($$

By substituting for terms into (4.8), using (4.9a)-(4.12), we obtain

$$\xi_{2a} = \in \bar{\xi}\left[\frac{2}{R} - \left(\frac{1+\cos t_0^{(2)}}{R^2-1}\right) + \frac{\delta\,\pi r_1}{\mathrm{R}}\right] + \in^2 \bar{\xi}\left[\left\{l_{13} + \delta\left(l_{17} + \bar{t}_0^{(2)}l_{15}\right)\right\}\right.$$
$$\left. + \bar{\xi}\left\{l_{14} + \delta\left(\frac{\pi r_1^2}{2Q^2R} + \bar{t}_0^{(2)}l_{16}\right)\right\}\right] + O\left(\in\delta^2\right) + O\left(\in^2\delta^2\right) \qquad (4.13)$$

The net maximum displacement ξ_m is the sum

$$\xi_m = \xi_{1a} + \xi_{2a} = \in C_1 + \in^2 C_2 + \cdots, C_1 = \bar{\xi}\left[\frac{2}{R} - \left(\frac{1+\cos t_0^{(2)}}{R^2-1}\right) + \frac{\delta\, r_1\pi}{R}\right] \qquad (4.14a)$$

$$C_2 = \bar{\xi}\left[l_{13} + \delta\left(l_{17} + \bar{t}_0^{(2)}l_{15}\right) + \bar{\xi}\left\{l_{14} + k_2Q^2S_3 + \delta\left(\frac{\pi\, r_1^2}{2Q^2R} + \bar{t}_0^{(2)}l_{16} + \bar{t}_0^{(1)}S_1 + 2S_2\right)\right\}\right] \qquad (4.14b)$$

According to Budiansky [2], the dynamic buckling load λ_D is obtained from the maximization

$$\frac{d\lambda}{d\xi_m} = 0 \qquad (4.15)$$

However the maximization (4.15) is preceded [8] by a reversal of the series (4.14a,b) in the following fashion

$$\in = d_1\xi_m + d_2\xi_m^2 + \cdots \qquad (4.16a)$$

By substituting for $\in$ from (4.14a) into (4.16a) and equating the coefficients of integral powers of $\in$, we get

$$d_1 = \frac{1}{C_1}, \; d_2 = -\frac{C_2}{C_1^3} \qquad (4.16b)$$

The maximization (4.15) is now easily accomplished through (4.16a) to give

$$\in = \frac{C_1}{4C_2} \qquad (4.17)$$

which is evaluated at $\lambda = \lambda_D$. On substituting for C_1 and C_2 in (4.17) from (4.14a, b), we get

$$\lambda_D = \frac{\frac{1}{4}\left(\frac{\omega_0}{\omega_2}\right)^2\left[\frac{2}{R^2} - \frac{\left(1+\cos t_0^{(2)}\right)}{R^2-1} + \frac{\delta\pi r_1}{R}\right]}{\left[\left\{l_{13} + \delta\left(l_{17} + \bar{t}_0^{(2)} l_{15}\right)\right\} + \bar{\xi}\left\{l_{14} + k_2Q^2S_3 + \delta\left(\frac{r_1^2\pi}{2Q^2R} + \bar{t}_0^{(2)} l_{16} + \bar{t}_0^{(1)} S_1 + 2S_2\right)\right\}\right]} \qquad (4.18)$$

5.0 Discussion of Result

The result (4.18) is asymptotic in nature and is applicable to cases of small values of $\bar{\xi}$, $\frac{\omega_2}{\omega_0}$ and $\frac{\omega_1}{\omega_0}$ .Since the right hand side of (4.18) is independent of the load parameter λ_D , the result so obtained is an algebraic formula that determines λ_D directly. All terms multiplying k_2 indicate the contribution to buckling of the quadratic term $k_2\xi_2^2$ in the original formulation (2.4). By extension, all terms (though absent) that multiply k_1 in (4.18) would similarly have indicated the contribution of the quadratic term $k_1\xi_1^2$.The consequences of neglecting the axisymmetric imperfection $\bar{\xi}_1$ are as follows :(i) We have automatically eliminated the contribution of the quadratic nonlinearity $k_1\xi_1^2$ which is in the shape of the neglected imperfection.(ii)The effect of the coupling between the buckling modes , that is the effect of the term $\xi_1\xi_2$, is similarly not felt.(iii) Neglecting only the axisymmetric imperfection $\bar{\xi}_1$ automatically results in the neglect of the quadratic term $k_1\xi_1^2$. Thus Danielson's assumption of neglecting both $k_1\xi_1^2$ and $\bar{\xi}_1$ is seen to be superfluous. From (4.18) we derive the result of the corresponding undamped step loading (δ=0) case as

$$\lambda_D = \frac{\frac{1}{4}\left(\frac{\omega_0}{\omega_2}\right)^2\left[\frac{2}{R} - \frac{\left(1+\cos\bar{t}_0^{(2)}\right)}{R^2-1}\right]}{l_{13} + \bar{\xi}\left(l_{14} + k_2Q^2S_3\right)} \qquad (5.1)$$

We can readily compute the result of the corresponding static loading case and obtain the static buckling load λ_s as

$$\left(1-\lambda_s\right)^2 = \frac{2\sqrt{3k_2}\,\lambda_s\bar{\xi}}{9} \qquad (5.2)$$

On eliminating the imperfection parameter $\bar{\xi}$ from (4.18) and (5.1) using (5.2), we get

$$\frac{\lambda_D}{\lambda_s} = \frac{\frac{1}{4}\left(\frac{\omega_0}{\omega_2}\right)^2\left[\frac{2}{R^2} - \frac{\left(1+\cos t_0^{(2)}\right)}{R^2-1} + \frac{\delta \pi r_1}{R}\right]}{\left[\lambda_s\left\{l_{13} + \delta\left(l_{17} + \bar{t}_0^{(2)} l_{15}\right)\right\} + \frac{9(1-\lambda_s)^2}{2\sqrt{3k_2}}\left\{l_{14} + k_2 Q^2 S_3 + \delta\left(\frac{\pi r_1^2}{2Q^2 R}\right) + \bar{t}_0^{(2)} l_{16} + \bar{t}_0^{(1)} S_1 + 2S_2\right\}\right]} \qquad (5.3)$$

$$\frac{\lambda_D}{\lambda_s} = \frac{\frac{1}{4}\left(\frac{\omega_0}{\omega_2}\right)^2\left[\frac{2}{R^2} - \frac{\left(1+\cos t_0^{(2)}\right)}{R^2-1}\right]}{\left[\lambda_s\left\{l_{13} + \delta\left(l_{17} + \bar{t}_0^{(2)} l_{15}\right)\right\} + \frac{9(1-\lambda_s)^2}{2\sqrt{3k_2}}\left(l_{14} + k_2 Q^2 S_3\right)\right]} \qquad (5.4)$$

Danielson's result for sep loading case (corresponding to (5.4)) is

$$\frac{\lambda_D}{\lambda_s} = \frac{\frac{1}{6}\left\{4 - \left(\frac{\omega_0}{\omega_2}\right)^2\right\}}{\frac{\lambda_s}{\lambda_c} + \left(\frac{32}{27}\right)^{\frac{1}{2}}\left(\frac{10}{9}\right)\left(\frac{\omega_1}{\omega_0}\right)\left(\frac{\omega_2}{\omega_0}\right)\left(1 - \frac{\lambda_s}{\lambda_c}\right)^2} \qquad (5.5)$$

Since Danielson's result was based on the concept of Mathieu-type of instability which we had earlier pointed out as not adequately depicting the phenomenon of dynamic buckling, we thus expect the result (5.4) to be a better approximation compared to (5.5).

6.0 Conclusion

We emphasize ,among other things ,that by neglecting the axisymmetric imperfection ,the coupling effect with other buckling modes of the buckling mode that is in the shape of the neglected imperfection ,is automatically nullified, hence the effect of the coupling term $\xi_1\xi_2$ is automatically nullified. This also nullifies Danielson's assumption that the effect of the coupling term $\xi_1\xi_2$ is more dominant compared to the effect the quadratic term $k_1\xi_1^2$ for conditions at the initial post buckling consideration. Infact the effects of both terms are certainly not felt, let alone the coupling term having a dominant effect over the quadratic nonlinearity.

References

[1] D. Danielson, Dynamic buckling loads of imperfection-sensitive structures from perturbation procedures, AIAA J. 7, 1506-1510 (1969).

[2] B. Budiansky, Dynamic buckling of structures: criteria and estimates, in, Dynamic stability of structures, Pergamon, New York , 1966.

[3] O. Aksogan and A.V. Sofiyev, Dynamic buckling of spherical shells with variable thickness subjected to a time dependent external pressure varying as a power function of time, J. of Sound and Vibration 254 (4), 693-703 (2002).

[4] A. Wang and W. Tian, Twin – characteristic parameter solution for dynamic buckling of columns under elastic compression-waves, Int. J. Solids Struct.39, 861-877 (2002a).

[5] A. Wang and W. Tian ,Characteristic-value analysis of plastic dynamic buckling of columns under elastoplastic compression waves, Int. J. Non- Linear Mech.35,615-628 (2002b).

[6] A. Wang and W. Tian, Twin- characteristic-parametric solution of axisymmetric dynamic plastic buckling for cylindrical shells under axial compression waves. Int. J. Solids Struct.40, 3157-3175 (2003).

[7] D. Karagiozova and N. Jones, Dynamic elasto-plastic buckling of cylindrical shells under axial impact, Int. J. Solids Struct. 37, 2005-2034 (2000).

[8] A.M. Ette, On a two-parameter buckling of a lightly damped spherical cap trapped by a step load, Journal of Nigerian Math. Society, 24,7-26 (2004) .