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Perturbed segmented domain collocation Tau-method for the numerical solution of Second Order Boundary Value problems

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Abstract

This paper concerns the numerical solution of second order boundary value problems using a Perturbed segmented domain collocation-Tau method. The entire interval for which the problem is defined is partitioned into two segments and the solution technique is demonstrated on each of the segments. The Chebyshev polynomials shifted as the case may be, into a given interval are used as a basis for a collocation solution via the perturbed collocation method for each segment. For a given problem two different solutions are obtained, which are valid for different intervals within the domain. Numerical examples are given to illustrate the efficiency, accuracy and computational cost of the method.

Keywords*:* Collocation, Segmented domain, Auxiliary equation, Partitioning, Residual equations

1.0 Introduction

This research paper has to do with the numerical solution of second order boundary problem of the form:

$$\alpha(x)\ \frac{d^2u(x)}{dx_2}\ +\ \Gamma(x)\,\frac{d\ u(x)}{dx}\ +\ \beta(x)\ u(x)\ =\ f(x) \tag{1.1}$$

Which is valid in some interval a $\leq\ x\ \leq$ b together with sufficient conditions imposed on the dependent variable at the two end points x = a and x = b.

Where x is the independent variable, u(x) is an unknown function, $\alpha(x)$, $\Gamma(x)$, $\beta(x)$ and $f(x)$ are known functions.. The physical applications of boundary value problem are found in (Ref. [2]). An active research work has extensively being carried out on this area, with a number of numerical methods of solution developed. For instances, [10] analyzed the problem using the methods of weighted residuals, [9] applied Tau method using some basis functions. As a way of enhancing the results, [2] demonstrated the use of Finite Element method (FEM) on it, with the prominent features of partitioning the domain into any number of elements.

In this research paper we have investigated the same problem using newly developed methods called Segmented Domain Collocation Tau method (SDCM) and Perturbed Segmented Domain Collocation Tau method (Perturbed SDCM). According to [11], the smaller the elements in FEM the better the accuracy of the solution. This basic fact is what is harnessed by these methods by dividing into two the entire domain of the problems solved.

By this work our aim is to compare the numerical results obtained using the method mentioned above with the exact solution of some problems in this class. This is carried out alongside with the numerical solution by some existing methods.

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2.0 Conversion techniques of Chebyshev polynomial

The Chebyshev polynomial of degree n valid in interval $-1 < x \leq 1$ is defined by;

$$T_n(x) = \cos\{n\cos^{-1} x\} \qquad (2.1)$$

The recurrence relation is given by; $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x); \quad n \geq 1$ (2.2)

For the sake of problems that exist within intervals other than [-1, 1]. A good $L\infty$ - Type approximation to a function $f(x)$ over [a, b] is applied. For the transformation of a variable, which maps; $a < X < b$ into $-1 \leq x \leq 1$.

Let $$X = \alpha x + \beta, \qquad (2.3)$$

where α and β are to be determined. $x \in [-1\ 1]$ and $X \in [a, b]$, then $a = -\alpha + \beta$ and $b = \alpha + \beta$

It follows that; $\alpha = \frac{b-a}{2}$ *and* $\beta = \frac{b+a}{2}$

Substituting this into (2.3) gives; $X = \frac{(b-a)x}{2} + \frac{(a+b)}{2} \Rightarrow 2X = (b-a)x + a + b$

or $$x = \frac{(a+b-2X)}{a-b} \qquad (2.4)$$

Thus substituting equation (2.4) into equations (2.1) and (2.2) we get the general formulae for conversion to any interval, where a and b are the bounds of the interval within which the new problem may fall.

3.0 Numerical solution techniques

Method 1
Segmented Domain Collocation Method (Sdcm)

This method is developed as an application of standard collocation method based on the principle of division of domain from Finite Element method (FEM).
In this method the interval $a \leq x \leq b$ for which the problem is defined is divided into two segments at the point x_c called the point of partition, over each of these segments, a trial solution $\bar{u}^{(1)}(x; a)$ and $\bar{u}^{(2)}(x; a)$ are formulated for segment one and two respectively. The following are the step-by-step approach towards the solution in this method.

Step 1
Partitioning of the domian into two segments

Suppose that the interval for which the given problem is $a \leq x \leq b$. This is divided into two equal portions, i.e.

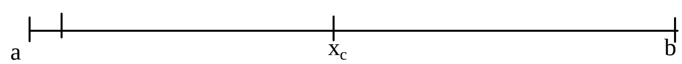

Figure 1.1

Note that; $$u(x_1) = a \text{ and } u(x_2) = b \qquad (3.1)$$
are the given boundary conditions.located between the two segments is the point x_c called the point of partition. By Applying x_c to $u^{(1)}(x)$ and $u^{(2)}(x)$ we generate **inter-segments conditions** respectively for segments one and two to be :

$$u^{(1)}(x_c) = c_1 \text{ and } u^{(2)}(x_c) = c_2 \qquad (3.2)$$

Step 2
Trial Solution Derivation

Let the trial solutions for both segment 1 and 2 be respectively denoted as: -

$$\bar{u}^{(1)}(x; a) = a_0\,\phi_0^{(1)}(x) + a_1\phi_1^{(1)}x + \cdots + a_N\,\phi_{N\ (x)}^{(1)} \qquad (3.3)$$

$$\bar{u}^{(2)}(x; a) = a_0\,\phi_0^{(2)}(x) + a_1\,\phi_1^{(2)}(x) + \cdots + a_N\,\phi_{N\ (x)}^{(2)} \qquad (3.4)$$

where x represents all the independent variables in the problem and functions $\phi_0(x), \phi_1(x), \cdots \phi_N(x)$ are known functions called trial functions (or sometimes basis or coordinate functions). The coefficients $a_1, a_2, \cdots, a_N$ are unknown parameters to be determined and are frequently called degrees of freedom (DOF) or sometimes generalized coordinates. See (Ref. [2]).

The construction of a trial solution consists of constructing expressions for each of the trial functions in terms of specific known functions.

As discussed in [2], from a practical standpoint, it is important to use functions that are algebraically as simple as possible and also easy to work with, because we frequently must calculate derivatives and integrals of the $\phi_i(x)$. Powers of x are certainly the easiest for these operations, so a logical choice for trial solutions used are the first few terms of a power series.i.e. polynomials of the form: $\bar{u}(x; a) = a_0 + a_1 x + \cdots + a_N x^N$ (3.5)

Specifically, $N = 3$ is used throughout this paper.

According to step 1, it is to be noted that for each segment, we have one Boundary Condition and one inter-segment condition, i.e. $u(x_1) = a$ and $u^{(1)}(x_c) = c_1$ for segment 1, and $u(x_2) = b$ and $u^{(2)}(x_c) = c_2$ for segment 2. For each segment, the two conditions are then imposed on the power series trial solution (3.5) and the unknown parameters a_i $i = 0, 1, \cdots N$ are then reduced in number by solving for two in terms of others and substituting their values into (3.5). This is done for each segment separately.

Step 3

Residual Equation Formualtion

Equation (1.1), which is the general form of second order Boundary value problem, is then written in the form:

$$\alpha(x)\frac{d^2u(x)}{dx^2} + \Gamma(x)\frac{du(x)}{dx} + \beta(x)u(x) - f(x) = 0 \qquad (3.6)$$

It is worthy of note that if equation (3.6) is valid for the entire interval a $\leq x \leq$ b, it is also valid for each segment of that interval.

According to [2], equation (3.6) implies that if the exact solution are substituted for $u(x)$ on the LHS, then the RHS would be identically zero over each of the segments. But if any other function such as the approximate trial solutions $u^{(1)}(x; a)$ *and* $u^{(2)}(x; a)$ are substituted for u(x), the result would be non-zero function called the residual error for segment one and two and denoted by $R^{(1)}(x; a)$ and $R^{(2)}(x; a)$ respectively, of which we have:

$$R^{(1)}(x;a) = \alpha(x)\frac{d^2\bar{u}^{(1)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(1)}(x;a)}{dx} + \beta(x)\bar{u}^{(1)}(x;a) - f(x) \neq 0 \qquad (3.7)$$

and

$$R^{(2)}(x;a) = \alpha(x)\frac{d^2\bar{u}^{(2)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(2)}(x;a)}{dx} + \beta(x)\bar{u}^{(2)}(x;a) - f(x) \neq 0 \qquad (3.8)$$

Where $u^{(1)}(x; a)$ *and* $u^{(2)}(x; a)$ are defined in step 2.

Step 4

Collocating each of the residual equations

The two residual equations in (3.7) and (3.8) are then collocated at points x_i called the points of collocation. This method required that for each unknown parameter a_i each of the residual equations, we choose a point x_i to be within the respective segment and not on the boundary, their location is not necessarily in any particular pattern but it might be reasonable to distribute them uniformly according to [2].

Collocation points for each segment are arrived at by the formula:

$$x_i = a + \frac{(b-a)(i)}{N}, \ i = 1, 2, \cdots N-1 \qquad (3.9)$$

where N is the degree of approximant, irrespective of the order of the differential equation being considered. At each point of collocation the residual equations (3.7) and (3.8), which are not equal to zero, are then forced to be zero, i.e. On segment 1, it gives:-

At $x = x_1; R^{(1)}(x_1; a) \Rightarrow 0$

$$\vdots \qquad \vdots \qquad (3.10)$$

At $x = x_{N-1}; R^{(1)}(x_{N-1}; a) \Rightarrow 0$

On segment 2, it gives: -

At $x = x_1; R^{(2)}(x_1; a) \Rightarrow 0$

$$\vdots \qquad \vdots \qquad (3.11)$$

At $x = x_{N-1}; R^{(2)}(x_{N-1}; a) \Rightarrow 0$

For a trial solution with N parameters, we therefore produce a system of $N-1$ linear equations for each segment.

Step 5
Auxiliary Equation and its Derivation

Because of the inter-segment condition, which is equal to unknown c_i (i = 1 2), present in the system of residual equations are c_1 and c_2 respectively for segment 1 and 2. Because of this, there is always a need for one more equation to be able to solve for the N number of unknowns in each segment. This equation is called **auxiliary equation**.

For segment 1, the auxiliary equation is gotten by applying the second boundary condition to segment 1's trial solution i.e. $\bar{u}^{(1)}(x_2;\ a) = b$. And for segment 2, we arrived at the auxiliary equation by applying the 1st boundary condition to segment 2's trial solution i.e. $\bar{u}^{(2)}(x_2;\ a) = a$.

Auxiliary equation gotten for each segment is then used in conjunction with the system of N - 1 linear equation for the same segment; the system of N equations is then solved simultaneously. The numerical values for a_i and c_i are then arrived at.

A note should be taken here because approximate solutions for each of the segments must be equal at the same point of x for which both are valid. This point is x_c which is the point of partition i.e.

$$\bar{u}^{(1)}(x_c) = \bar{u}^{(2)}(x_c) = c_1 = c_2 \qquad (3.12)$$

If (3.12) is satisfied, the parameters arrived at are substituted into the trial solutions for each segments and that gives the approximate solution for the problem at that interval. But if $c_1 \neq c_2$, check out for the c that produces optimum solution at point x_c for which both equations are valid.

To check for this, substitute parameters for each of the segments into their respective residual equation. Then choose the c which produces the residual error with smallest modulus at point x_c and use it for the two c's. The parameters a_i arrived at together with the chosen c are then substituted into the trial solution for each segment and that gives solution for the problem at the interval for which that segment is defined.

Method 2
Perturbed Segmented Domain Collocation Tau Method (Perturbed Sdcm)

The development of this method is the same with that of method 1, up to step 3 where we have equation (3.7) and (3.8) which can respectively, be written as:

$$R^{(1)}(x;\ a) = \alpha(x)\,\frac{d^2\ \bar{u}^{(1)}\ (x;\ a)}{dx^2} + \Gamma(x)\,\frac{d\ \bar{u}^{(1)}\ (x;\ a)}{dx} + \beta\ \bar{u}^{(1)}\ (x;\ a) \neq f(x) \qquad (3.14)$$

and

$$R^{(2)}(x;\ a) = \alpha(x)\,\frac{d^2\ \bar{u}^{(2)}\ (x;\ a)}{dx^2} + \Gamma(x)\,\frac{d\ \bar{u}^{(2)}\ (x;\ a)}{dx} + \beta\ \bar{u}^{(2)}\ (x;\ a) \neq f(x) \qquad (3.15)$$

Step 4
Perturbing and Collocating the Residual Equations

According to [2], the meaning of equations (3.14) and (3.15) is that when the approximate trial solution $\bar{u}^{(1)}(x;\ a)$ and $\bar{u}^{(2)}(x;\ a)$ or any other function other than the exact solution $u(x)$ are substituted into the given ODE, the LHS will not equal to the RHS again at it was before.

According to the idea of the Tau-method, as conceived by Lanczos [3] is the addition to equations (3.14) and (3.15) of a small perturbation term $H_n(x)$ which causes (3.14) and (3.15) to respectively become:

$$\alpha(x)\,\frac{d^2\,\bar{u}^{(1)}(x;a)}{dx^2} + \Gamma(x)\,\frac{d\,\bar{u}^{(1)}(x;a)}{dx} + \beta\,\bar{u}^{(1)}(x;a) = f(x) + H_n(x) \qquad (3.16)$$

$$\alpha(x)\,\frac{d^2\,\bar{u}^{(2)}(x;a)}{dx^2} + \Gamma(x)\,\frac{d\,\bar{u}^{(2)}(x;a)}{dx} + \beta\,\bar{u}^{(2)}(x;a) = f(x) + H_n(x) \qquad (3.17)$$

The perturbation term $H_n(x)$ used in this work is of the form: $H_n(x) = \tau_1 T_n(x) + \tau_2 T_{n-1}(x)$, where n the order of approximant and chosen to be 4 τ_I and τ_2 are Tau-parameters to be determined. $T_n(x)$ is defined in equation (2.1).

For problems that are existing in domain other than interval $-1 \le x \le 1$, a conversion technique discussed in 2 is employed to arrive at the shifted chebyshev polynomials for that interval, equation (3.16) and (3.17) are then collocated at some selected points called collocation points x_i ,which are chosen to be within each segment by the formula: $x_i = a + \frac{(b-a)(i)}{N+2},\ i = 1,\ 2, \cdots N+1$

where a and b are the bounds for each segment.

The system of N + 1 linear equations produced in the process of collocating each of equation (20) and (21), in conjunction with the auxiliary equation gotten in the same way as that of method one, are solved simultaneously to arrive at numerical values for the unknowns a_i and c_i . These values are then substituted into our approximate trial solutions.

3.0 Numerical examples

Example 1

Solve the Boundary Value Problem whose governing equation is: $(x+1)\frac{d^2u}{dx^2} + \frac{du}{dx} = 0$

within the interval $1 \le x \le 2$, and with the boundary condition $u(1) = 1$ and $\left[-(x+1)\frac{du}{dx}\right]_{x=2} = 1$

the exact solution is; $u(x) = 1 - \ln\left[{(x+1)}/{2}\right]$

Example 2

Solve the Boundary Value Problem: $x^3\frac{d^2u}{dx^2} + x^2\frac{du}{dx} - 2 = 0$, within the interval $1 \le x \le 2$ with boundary conditions are: $u(1) = 2$ and $\left[-x\frac{du}{dx}\right]_{x=2} = \frac{1}{2}$. The analytical solution is: $u(x) = \frac{2}{x} + \frac{\ln x}{2}$

Example 3

Solve the Boundary Value Problem:

$\frac{12x^2 d^2u}{dx^2} + \frac{24x\,d\,u}{dx} = -30x^4 + 204x^3 - 351x^2 + 110x,\ 0 \le x \le 1$, with the boundary conditions $u(0) = 1$ and u(1) = 2, the exact solution is:- $u(x) = \frac{1}{24}\left(-3x^4 + 34x^3 - 117x^2 + 110x + 24\right)$.

Table 1: Errors for Example 1

x	Standard Collocation	Perturbed Collocation	SDCM	Perturbed SDCM
1.0	0	0	0	0
1.1	4.0×10^{-4}	3.20×10^{-3}	5.0×10^{-4}	1.0×10^{-4}
1.2	7.0×10^{-4}	5.90×10^{-3}	9.0×10^{-4}	2.0×10^{-4}
1.3	1.0×10^{-3}	8.20×10^{-3}	1.3×10^{-3}	3.0×10^{-4}
1.4	1.1×10^{-3}	1.00×10^{-2}	1.5×10^{-3}	3.0×10^{-4}
1.5	1.3×10^{-3}	1.16×10^{-2}	1.7×10^{-3}	4.0×10^{-4}
1.6	1.6×10^{-3}	1.29×10^{-2}	1.9×10^{-3}	4.0×10^{-4}
1.7	1.8×10^{-3}	1.39×10^{-2}	1.7×10^{-3}	3.0×10^{-4}
1.8	2.0×10^{-3}	1.46×10^{-2}	1.5×10^{-3}	3.0×10^{-4}
1.9	2.2×10^{-3}	1.51×10^{-2}	9.0×10^{-4}	3.0×10^{-4}
	2.3×10^{-3}	1.53×10^{-2}	1.0×10^{-2}	3.0×10^{-4}

Table 2: Errors for example 2

x	Standard Collocation	Perturbed Collocation	SDCM	Perturbed SDCM
1.0	0	0	0	0
1.1	1.97×10^{-2}	1.35×10^{-2}	2.64×10^{-2}	9.4×10^{-2}
1.2	3.17×10^{-2}	2.08×10^{-2}	4.36×10^{-2}	1.47×10^{-2}
1.3	4.02×10^{-2}	2.61×10^{-2}	5.47×10^{-2}	1.86×10^{-2}
1.4	4.76×10^{-2}	3.13×10^{-2}	6.03×10^{-2}	2.2×10^{-2}
1.5	5.49×10^{-2}	3.74×10^{-2}	6.03×10^{-2}	2.45×10^{-2}
1.6	6.26×10^{-2}	4.45×10^{-2}	6.11×10^{-2}	2.25×10^{-2}
1.7	7.02×10^{-2}	5.21×10^{-2}	6.15×10^{-2}	2.11×10^{-2}
1.8	7.71×10^{-2}	5.92×10^{-2}	6.20×10^{-2}	2.05×10^{-2}
1.9	8.21×10^{-2}	6.45×10^{-2}	6.24×10^{-2}	2.05×10^{-2}
	8.41×10^{-2}	6.67×10^{-2}	6.27×10^{-2}	2.06×10^{-2}

Table 3: Example Errors For 3

x	Standard Collocation	Perturbed Collocation	SDCM	Perturbed SDCM
0	0	0	0	0
0.1	2.5×10^{-3}	7.0×10^{-3}	3.1×10^{-3}	6.1×10^{-3}
0.2	3.2×10^{-3}	9.5×10^{-3}	4.3×10^{-3}	8.4×10^{-3}
0.3	3.2×10^{-3}	9.0×10^{-3}	4.6×10^{-3}	8.2×10^{-3}
0.4	3.0×10^{-3}	7.0×10^{-3}	4.7×10^{-3}	6.8×10^{-3}
0.5	3.2×10^{-3}	4.6×10^{-3}	4.9×10^{-3}	4.8×10^{-3}
0.6	3.7×10^{-3}	2.3×10^{-3}	3.2×10^{-3}	2.5×10^{-3}
0.7	4.3×10^{-3}	7.0×10^{-4}	1.8×10^{-3}	8.0×10^{-4}
0.8	4.5×10^{-3}	0	1.0×10^{-3}	0
0.9	3.5×10^{-3}	1.0×10^{-4}	6.0×10^{-4}	1.0×10^{-4}
1.0	0	0	0	0

5.0 Conclusion

Tables 1-3 show the numerical solutions in terms of the maximum errors obtained for the second order boundary value problems considered in this paper. It is clearly observed that, of the four methods used, the Perturbed Domain Collocation-tau Method (Perturbed SDCM) produced results with least error than the other three methods. Also it is noticed that as N increases, the algebraic system of linear equations to be solved also increases. For example, when N = 3, the Perturbed Domain collocation method resulted to five algebraic equations while the remaining methods resulted to two or four algebraic linear equations. The extra work done in perturbed SDCM is being compensated for, in terms of the accuracy obtained.

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