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A Continuous formulation of some classical initial value volvers by non-Perturbed multistep collocation approach using Chebyshev polynomials as basis functions

R. B. Adeniyi[+], E. O. Adeyefa and M. O. Alabi[*]
Department of Mathematics, University of Ilorin, Ilorin, Nigeria.

Abstract

This paper is concerned with the construction of some classes of multistep methods for the numerical integration of initial value problems in ordinary differential equations. For this purpose we employ the Chebyshev polynomials as basis function in a non-perturbed collocation approach. The continuous schemes thus obtained yield four classes of initial value solvers namely the Optimal order methods, the Adams-Bashforth methods, the Adams-Moulton methods and the Backward differentiation formulae at appropriate grid points. A theorem in support of the accuracy of the continuous schemes is also established.

1.0 Introduction

Recently research on the formulation of initial value solvers as continuous approximation schemes for the integration of ordinary differential equations has progressed tremendously and reported by quite a number of authors, some of whom have employed either the monomials $\{x^r\}$, $r = 0(1)n$, as in [4], [6], [18], [14] or the so-called canonical polynomials $\{Q_r(x)\}$, $r = 0(1)n$, as in [1], [2], [4] of the Lanczos method [9], [11] as basis functions. Some of these earlier works will be reviewed briefly in Section 2 of this paper.

It is, however, desirable to consider technique which are based on the well-behaved Chebyshev polynomials $\{T_r(x\}$, r = 0(1)n, which are known to oscillate with equal amplitude in the entire range of definition, thus ensuring even distribution of error, in contrast with the popular Taylor series which diminishes in accuracy as one moves further away from the origin/centre. As will be shown later in this work, this characteristic feature of $T_r(x)$ is also exhibited by the continuous approximant which we shall propose here and which we have formulated based on $T_r(x)$.

Consequently, for the initial value problem (IVP)

$$y'(x) = f(x,y(x)), \qquad a \le x \le b \qquad (1.1a)$$

$$y(a) = y_a \qquad (1.1b)$$

we shall seek an approximation of the form

$$Y(x) = \sum_{r=0}^{n} a_r T_r(x) \cong y(x) \qquad (1.2)$$

over the segmented interval. $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ and where the points $\{x_r\}$, $r = 0(1)n$, have uniform spacing $h = (b-a)/n$ such that $x_r = x_0 + rh$. For convenience and without loss of generality we shall let $a = 0$ in the differential system (1.1) so that $x_r = rh$ and since by appropriate transformation (1.1) may be redefined for the interval $[o,b]$. This will be our concern in Section 3 of the paper.

+Corresponding author concerning this paper.
***Author is of the Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, Ogbomoso, Nigeria.**

As stated earlier, Section 2 of the paper focuses on a brief review of earlier work on the subject. In Section 4, we shall provide some numerical evidences in support of this work. Finally we shall end the paper with some concluding remarks in Section 5.

2.0 Review of earlier works

We shall briefly discuss here a method, which is based on the power series in a non-perturbed collocation technique, and another method based on the canonical polynomials in a perturbed collocation procedure, as classical cases of earlier works on the subject matter**.**

2.1 Methods based on Power Series

We review here the work reported in [4] and [10] for a second degree approximant of y(x) as a particular case. To this end we insert

$$Y(x) = a_0 + a_1(x) + a_2(x) \cong y(x), \quad x_k \le x \le x_{k+1} \tag{2.1}$$

in (1.1a) and then collocate the resulting equation at x_k and x_{k+1} as well as interpolate (2.1) at x_{k+1} to get the system

$$\begin{pmatrix} 1 & x_{k+1} & x_{k+1}^2 \\ 0 & 1 & 2x_k \\ 0 & 1 & 2x_{k+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{k+1} \\ f_k \\ f_{k+1} \end{pmatrix} \tag{2.2}$$

We solve this system and subsequently obtain from (2.1) the continuous scheme

$$Y(x) = y_{k+1} + \frac{(x-x_k)^2}{2h} f_k + \frac{(x-x_{k+1})}{2h}(x + x_{k+1} - 2x_k) f_{k+1} \tag{2.3}$$

At the grid point x_{k+2}, this yields the two-step Adams-Bashforth explicit scheme

$$Y_{k+2} = y_k + \frac{h}{2}(3f_{k+1} - f_k) \tag{2.4}$$

2.2 Methods Based on Canonical Polynomials

We review here a perturbed method reported in [ ] and which is based on the canonical polynomials $\{Q_r(x)\}$, defined by $LQ_r(x) = x^r, \quad r \ge 0$, where L is a linear differential operator associated with (1.1a).. For this purpose we seek

$$Y(x) = \sum_{r=0}^{n} a_r Q_r(x) \cong y(x), \quad x_k \le x \le x_{k+n} \tag{2.5}$$

where $Q_r(x)$ is given by

$$Q_r(x) = x^r - rQ_{r-1}(x) \tag{2.6}$$

And which satisfies exactly the perturbed problem

$$Y'(x) = f(x,Y(x)) + \tau P_n(x), \quad Y(x_k) = Y_k, \qquad x_k \le x \le x_{k+n} \tag{2.7}$$

The free τ parameter in (2.6) is to be determined along with the a_r's and $P_n(x)$ is the nth degree Legendre polynomial valid in (x_k, x_{k+n}).. We collocate (2.6) at x_{k+r}, $r = 0(1)n$ and interpolate (2.5) at x_k to get the system.

$$\begin{pmatrix} 1 & x_k - 1 & x_k^2 - 2x_k + 2 & \\ 0 & 1 & 2(x_k - 1) & 1 \\ 0 & 1 & 2(x_{k+1} - 1) & \frac{1}{2} \\ 0 & 1 & 2(x_{k+2} - 1) & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \tau \end{pmatrix} = \begin{pmatrix} Y_k \\ f_k \\ f_{k+1} \\ f_{k+2} \end{pmatrix}$$

From this and (2.5) we get a continuous scheme which at the grid point x_{k+2} yield the Simpson's scheme

$$P_k(x) = \sum_{r=0}^{k} C_r^{(k)} x^r,$$

3.0 Methods Based On Chebyshev Polynomials

We shall now consider here a modification of the method of the preceding section by seeking an approximation $Y(x) = \sum_{r=0}^{M} a_r T_r(x) \cong y(x), \quad x_k \le x \le x_{k+p}$

over each of the sub-interval $[x_k, x_{k+p}]$ of $[a,b]$ where M and P will be appropriately chosen so as to derive four classes of multistep methods .

So then we shall solve the non-perturbed IVP:

$$Y'(x) = f(x, Y(x)), \; x_k \le x \le x_{k+p} \qquad (3.1a)$$

$$Y(x_k) = Y_k \qquad (3.1b)$$

$$Y(x) = \sum_{r=0}^{M} a_r T_r(x) \cong y(x), \quad x_k \le x \le x_{k+p} = \sum_{r=0}^{M} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \qquad (3.6)$$

The rth degree Chebyshev polynomial $T_r(x)$ in (3.2) is defined as

$$T_r(x) = \cos\left[r\cos^{-1}\left\{\frac{2x-(b-a)}{b-a}\right\}\right] \equiv \sum_{m=0}^{r} C_m^{(r)} x^m, \quad a \le x \le b \qquad (3.3)$$

and satisfies the recurrence relation

$$T_{r+1}(x) = 2\left(\frac{2x-b-1}{b-a}\right)T_r(x) - T_{r-1}(x), \quad r \ge 1 T_0(x) = 1, \quad T_1(x) = (2x-b-a)/(b-a) \qquad (3.4)$$

Of all monomials in $[a,b]$, $\left(C_r^{(r)}\right)^{-1} T_r(x)$ has the least maximum magnitude. Hence the desirability of $T_r(x)$ in this work.

3.1 Optimal Order Methods

We derive here continuous formulation of two optimal order initial value solvers for (1.1) by choosing $M = n+1$ and $P = n$ in (3.1) – (3.2), that is

$$Y'(x) = f(x, Y(x)), \; x_k \le x \le x_{k+n} \qquad (3.5a)$$

$$Y(x_k) = Y_k \qquad (3.5b)$$

$$Y(x) = \sum_{r=0}^{n+1} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right)$$

We shall collocate (3.5a) at the $(n+1)$ points x_{k+r}, $r = 0(1)n$ and interpolate (3.6) at x_k to give the $(n+2)$ equation for the unique determination of a_r, $r = 0(1)n+1$, in (3.6). So doing we have

$$Y'(x_{k+r}) = f_{k+r}, \quad r = 0(1)n$$

$$Y(x_k) = Y_k \qquad (3.7)$$

Let us now consider specific cases of (3.6) – (3.7).

3.1.1 A One-Step Method *(n = 1)*

Suppose $n = 1$ in (3.7) so that

$$Y'(x_k) = f_k$$

$$Y'(x_{k+1}) = f_{k+1}$$

$$Y(x_k) = Y_k$$

That is

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -8 \\ 0 & 2 & 8 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \tau \end{pmatrix} = \begin{pmatrix} Y_k \\ hf_k \\ hf_{k+1} \end{pmatrix}$$

We solve this system to get

$$a_0 = Y_k + \frac{h}{16}(5f_k + 3f_k), \quad a_1 = \frac{h}{4}(f_k + f_{k+1}), \quad a_2 = \frac{h}{16}(f_{k+1} - f_k)$$

We now insert these in (3.6) with n = 1 to get the continuous scheme

$$Y(x) = Y_k + h[\beta_0(x) f_k + \beta_1(x) f_{k+1}] \tag{3.8}$$

where

$$\left.\begin{aligned} \beta_0(x) &= \frac{-(x - x_k)^2}{2h^2} + \frac{(x - x_k)}{h} \\ \beta_1(x) &= \frac{(x - x_k)^2}{2h^2} \end{aligned}\right\}$$

We evaluate (3.8) at x_{k+1} to obtain the optimal order one-step method

$$Y_{k+1} = Y_k + \frac{h}{2}(f_{k+1} - f_k) \tag{3.9}$$

otherwise called the Trapezoidal method.

3.1.2 A Two-Step Method *(n = 2)*

For a two-step continuous formulation we now consider (3.6) – (3.7) with $n = 2$ to have

$$Y'(x_k) = f_k$$
$$Y'(x_{k+1}) = f_{k+1}$$
$$Y'(x_{k+2}) = f_{k+2}$$
$$Y(x_k) = Y_k$$

This leads to the system

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -4 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_i \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \end{pmatrix}$$

whose solution yields the values

$$a_0 = Y_k + \frac{h}{16}(5f_k + 3f_k),$$

$$a_1 = \frac{h}{8}(f_k + 6f_{k+1} + f_{k+2}),$$

$$a_2 = \frac{h}{8}(f_{k+2} - f_k)$$

$$a_3 = \frac{h}{24}(f_k - 2f_{k+1} + f_{k+2}).$$

We insert these in (3.6) with $n = 2$ to get the continuous scheme

$$Y(x) = Y_k + h[\beta_0(x) f_k + \beta_1(x) f_{k+1} + \beta_2(x) f_{k+2}] \tag{3.10}$$

where

$$\left.\begin{aligned} \beta_0(x) &= \frac{(x - x_k)^3}{6h^3} + \frac{3(x - x_k)^2}{4h^2} + \frac{(x - x_k)}{h} \\ \beta_1(x) &= -\frac{(x - x_k)^3}{3h^3} + \frac{(x - x_k)^2}{h} \\ \beta_2(x) &= \frac{(x - x_k)^3}{h^2} - \frac{(x - x_k)^2}{4h} \end{aligned}\right\}$$

At the grid point x_{k+2}, (3.10) gives the discrete form

$$Y_{k+2} = \frac{h}{3}\left(f_k + 4f_{k+1} + f_{k+2}\right) \qquad (3.11)$$

which is the Simpson's optimal order two-step method.

The procedure may be continued to obtain higher order methods. We shall now proceed to the Adams-Bashforth methods.

3.2 The Adams-Bashforth Method

We construct here some continuous schemes which yield Adams-Bashforth explicit methods at the grid points by letting $M = P = n$ in (3.1) – (3.2) to have

$$Y'(x) = f(x, Y(x)),\ x_k \le x \le x_{k+n} \qquad (3.12a)$$

$$Y(x_k) = Y_k \qquad (3.12b)$$

$$Y(x) = \sum_{r=0}^{n} a_r T_r(x), \quad x_k \le x \le x_{k+n}$$

$$Y(x) = \sum_{r=0}^{n} a_r T_r\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \qquad (3.13)$$

For the unique determination of the $(n+1)$ – coefficients a_r, $r = 0(1)n$, in (3.13) we shall collocate (3.12) at the n points x_{k+r}, $r = 0(1)n-1$, and interpolate (3.13) at x_{k+n-1} to give

$$\left.\begin{array}{l} Y'(x_{k+r}) = f_{k+r}, \quad r = 0(1)n-1 \\ Y(x_{k+n-1}) = Y_{k+n-1} \end{array}\right\} \qquad (3.14)$$

We shall now consider specific cases of (3.13) – (3.14).

3.2.1 A One-Step Method

When n = 1 in (3.13) – (3.14) we have

$$Y'(x_k) = f_k$$
$$Y(x_k) = Y_k$$

That is

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} Y_k \\ hf_k \end{pmatrix}$$

which gives the values $a_0 = Y_k + \frac{1}{2}hf_k, \quad a_1 = \frac{1}{2}hf_k$. We insert these in (3.13) with n = 1 to have the continuous scheme

$$Y(x) = Y_k + h\beta_0(x) f_k \qquad (3.15)$$

where $\beta_0(x) = \frac{x - x_k}{h}$, At the grid point x_{k+1} this yields the one-step Adams-Bashforth explicit method

$$Y_{k+1} = Y_k + hf_k \qquad (3.16)$$

From (3.16) we obtain f_k for our proposed continuous scheme (3.15).

3.2.2 A Two-Step Method

When $n = 2$ in (3.13) – (3.14) we have

$$Y'(x_k) = f_k$$
$$Y'(x_{k+1}) = f_{k+1}$$
$$Y(x_{k+1}) = Y_{k+1}$$

This leads to the system

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Y_{k+1} \\ hf_k \\ hf_{k+1} \end{pmatrix}$$

whose solution is

$$a_0 = Y_{k+1} + \frac{h}{4}\left(f_{k+1} - f_k\right), \quad a_1 = hf_{k+1}, \; a_2 = \frac{h}{4}\left(f_{k+1} - f_k\right)$$

These together with (3.13) for *n = 2* gives the continuous scheme

$$Y(x) = Y_{k+1} + h\left[\beta_0(x)\, f_k + \beta_1(x)\, f_{k+1}\right] \qquad (3.17)$$

where

$$\left.\begin{aligned} \beta_0(x) &= -\frac{(x-x_k)^2}{2h^2} + \frac{(x-x_k)}{h} - \frac{1}{2} \\ \beta_1(x) &= \frac{(x-x_k)^2}{2h^2} - \frac{1}{2} \end{aligned}\right\}$$

From this we get the two-step Adams-Bashforth method

$$Y_{k+2} = Y_{k+1} + \frac{h}{2}\left(3 f_{k+1} - f_k\right) \qquad (3.18a)$$

3.2.3 A Three-Step Method

For a three-step method (n = 3), from (3.13) – (3.14) we have

$Y'(x_k) = f_k$,, $Y'(x_{k+1}) = f_{k+1}$

$Y'(x_{k+2}) = f_{k+2}$,, $Y(x_{k+2}) = Y_{k+2}$

These leads to the system

$$\begin{pmatrix} 27 & 9 & -21 & -23 \\ 0 & 2 & -8 & 18 \\ 0 & 6 & -8 & -10 \\ 0 & 6 & 8 & -10 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 27Y_{k+2} \\ 3hf_k \\ 9hf_{k+1} \\ 9hf_{k+2} \end{pmatrix}$$

We solve this to get

$$a_0 = Y_{k+2} + \frac{h}{48}\left(2 f_k - 37 f_{k+1} + 11 f_{k+2}\right),$$

$$a_1 = \frac{3h}{64}\left(5 f_k - f_{k+1} + 21 f_{k+2}\right),$$

$$a_2 = \frac{9h}{16}\left(f_{k+2} - f_{k+1}\right),$$

$$a_3 = \frac{9h}{64}\left(f_k - 2 f_{k+1} + f_{k+2}\right)$$

These with (3.13) for n = 3 yields the continuous scheme

$$Y(x) = Y_{k+2} + h\left[\beta_0(x)\, f_k + \beta_1(x)\, f_{k+1} + \beta_2(x)\, f_{k+2}\right] \qquad (3.18b)$$

where

$$\left.\begin{aligned} \beta_0(x) &= \frac{(x-x_k)^3}{6h^3} + \frac{3(x-x_k)^2}{4h^2} + \frac{(x-x_k)}{h} - \frac{1}{2} \\ \beta_1(x) &= \frac{(x-x_k)^3}{3h^3} + \frac{(x-x_k)^2}{h^2} + \frac{9(x-x_k)}{8h} - \frac{4}{3} \\ \beta_2(x) &= \frac{(x-x_k)^3}{6h^3} - \frac{(x-x_k)^2}{4h} - \frac{1}{3} \end{aligned}\right\}$$

At the grid point x_{k+3} this yields the three-step Adams-Bashforth method

$$Y_{k+3} = Y_{k+2} + \frac{h}{12}\left(5 f_k - 16 f_{k+1} + 23 f_{k+2}\right) \qquad (3.19)$$

We proceed this way to obtain methods of higher step numbers.

3.3 The Adams-Moulton Methods

For a continuous formulation of the Adams-Moulton methods we set $M = n + 1$ and $P = n$ in (3.1) – (3.2) and thus have

$$Y'(x) = f(x, Y(x)),\ x_k \le x \le x_{k+n} \tag{3.20a}$$

$$Y(x_k) = Y_k \tag{3.20b}$$

$$Y(x) = \sum_{r=0}^{n+1} a_r T_r\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right), \quad x_k \le x \le x_{k+n} \tag{3.21}$$

We collocate (3.20) at x_{k+r}, $r = 0(1)n$ and interpolate (3.21) at x_{k+n-1} for the determination of the $(n + 2)$ coefficients a_r, $r = 0(1)n+1$ in (3.21). So then we have

$$\left.\begin{aligned} Y'(x_{k+r}) &= f_{k+r}, \quad r = 0(1)n \\ Y(x_{k+n-1}) &= Y_{k+n-1} \end{aligned}\right\} \tag{3.22}$$

Let us now demonstrate the techniques for specific cases of (3.21) – (3.22).

3.3.1 A One Step Method

When $n = 1$ in (3.21) – (3.22) we have

$$Y'(x_k) = f_k,$$
$$Y'(x_{k+1}) = f_{k+1}$$
$$Y(x_k) = Y_k$$

From these we get the linear algebraic system

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -8 \\ 0 & 2 & 8 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Y_k \\ hf_k \\ hf_{k+1} \end{pmatrix}$$

and from which we obtain the values

$$a_0 = Y_k + \frac{h}{16}(3f_{k+1} + 5f_k)$$

$$a_1 = \frac{h}{4}(f_{k+1} + f_k)$$

$$a_2 = \frac{h}{16}(f_{k+1} - f_k)$$

We insert these in (3.21) for n = 1 to have the continuous formulation

$$Y(x) = Y_k + h[\beta_0(x) f_k + \beta_1(x) f_{k+1}] \tag{3.23}$$

where

$$\left.\begin{aligned} \beta_0(x) &= -\frac{(x - x_k)^2}{2h^2} + \frac{(x - x_k)}{h} \\ \beta_1(x) &= \frac{(x - x_k)^2}{2h^2} \end{aligned}\right\}$$

The corresponding discrete form is the one-step Adams-Moulton scheme

$$Y_{k+1} = Y_k + \frac{h}{2}(f_k + f_{k+1}) \tag{3.24}$$

also known as the Trapezoidal method.

3.3.2 A Two-Step Method

Let $n = 2$ in (3.21) – (3.22) so that

$$Y'(x_k) = f_k,\ Y'(x_{k+1}) = f_{k+1}$$
$$Y'(x_{k+2}) = f_{k+2},\ Y(x_{k+1}) = Y_{k+1}$$

From these we get the linear algebraic system

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -4 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 4 & 9 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} Y_{k+1} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \end{pmatrix}$$

which yields the values

$$a_0 = Y_{k+1} + \frac{h}{8}\left(f_{k+2} - f_k\right)$$

$$a_1 = \frac{h}{8}\left(f_{k+2} + 6f_{k+1} + f_k\right)$$

$$a_2 = \frac{h}{8}\left(f_{k+2} - f_k\right)$$

$$a_3 = \frac{h}{24}\left(f_{k+2} - 2f_{k+1} + f_k\right)$$

We insert these in (3.21) with $n = 2$ and so obtain the continuous scheme

$$Y(x) = Y_{k+1} + h\left[\beta_0(x) f_k + \beta_1(x) f_{k+1} + \beta_2(x) f_{k+2}\right] \tag{3.25}$$

where in this case

$$\left.\begin{aligned} \beta_0(x) &= \frac{(x-x_k)^3}{6h^3} - \frac{3(x-x_k)^2}{h^2} + \frac{(x-x_k)}{h} - \frac{5}{12} \\ \beta_1(x) &= -\frac{(x-x_k)^3}{6h^3} + \frac{(x-x_k)^2}{6h^2} + \frac{3(x-x_k)}{8h} - \frac{17}{24} \\ \beta_2(x) &= \frac{(x-x_k)^3}{6h^3} - \frac{(x-x_k)^2}{4h^2} + \frac{1}{12} \end{aligned}\right\}$$

From this we get the two-step Adams-Moulton implicit method

$$Y_{k+2} = Y_{k+1} + \frac{h}{12}\left(5f_{k+2} + 8f_{k+1} - f_k\right) \tag{3.26}$$

From (3.26 we determine f_{k+2} for our proposed continuous scheme (3.22).

3.3.3 A Three-Step Method

For a three-step method we set $n = 3$ in (3.21) – (3.22) and thus have

$$Y'(x_k) = f_k,\ Y'(x_{k+1}) = f_{k+1},\ Y'(x_{k+2}) = f_{k+2}$$
$$Y'(x_{k+3}) = f_{k+3},\ Y(x_{k+2}) = Y_{k+2}$$

These lead to the system

$$\begin{pmatrix} 81 & 27 & -63 & -69 & 17 \\ 0 & 2 & -8 & 18 & -32 \\ 0 & 162 & -72 & -90 & 224 \\ 0 & 162 & 72 & -90 & -224 \\ 0 & 2 & 8 & 18 & 32 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 81Y_{k+2} \\ 3hf_k \\ 81hf_{k+1} \\ 81hf_{k+2} \\ 3hf_{k+3} \end{pmatrix}$$

the solution of which is

$$a_0 = Y_{k+2} + \frac{h}{3072}\left(115f_k + 1639f_{k+1} + 25f_{k+2} - 243f_{k+3}\right)$$

$$a_1 = \frac{3h}{128}\left(5f_k + 27f_{k+1} + 5f_{k+3}\right)$$

$$a_2 = \frac{3h}{256}\left(7f_{k+3} + 27f_{k+2} - 27f_{k+1} - 5f_k\right)$$

$$a_3 = \frac{9h}{128}\left(f_k - f_{k+1} - f_{k+2} + f_{k+3}\right)$$

$$a_4 = \frac{27h}{1024}\left(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k\right)$$

We insert these in (3.21) with $n = 3$ to have the continuous scheme

$$Y(x) = Y_k + h\left[\beta_0(x) f_k + \beta_1(x) f_{k+1} + \beta_2(x) f_{k+2} + \beta_3(x) f_{k+3}\right] \qquad (3.27)$$

where

$$\beta_0(x) = -\frac{(x-x_k)^4}{24h^4} + \frac{(x-x_k)^3}{3h^3} - \frac{-11(x-x_k)^2}{12h^2} + \frac{19(x-x_k)}{16h} + \frac{1}{24}$$

$$\beta_1(x) = \frac{(x-x_k)^4}{8h^4} - \frac{5(x-x_k)^3}{6h^3} + \frac{3(x-x_k)^2}{2h^2} - \frac{9(x-x_k)}{16h} - \frac{59}{24}$$

$$\beta_2(x) = -\frac{(x-x_k)^4}{24h^4} + \frac{2(x-x_k)^3}{3h^3} - \frac{-3(x-x_k)^2}{12h^2} + \frac{9(x-x_k)}{16h} + \frac{19}{24}$$

$$\beta_3(x) = \frac{(x-x_k)^4}{24h^4} - \frac{(x-x_k)^3}{6h^3} + \frac{(x-x_k)^2}{6h^2} - \frac{3(x-x_k)}{16h} - \frac{3}{8}$$

and which at the grid point x_{k+3} yields the three-step Adams-Moulton method

$$Y_{k+3} = Y_{k+2} + \frac{h}{24}\left(f_k - 5f_{k+1} + 19f_{k+2} + 9f_{k+3}\right) \qquad (3.28)$$

3.4 The backward differentiation Formulae

We shall now derive the continuous formulation of the backward differentiation formulae (BDFs) by letting $M = P = n$ in (3.1) – (3.2) and thus we have

$$Y'(x) = f(x, Y(x)),\ x_k \le x \le x_{k+n} \qquad (3.29a)$$

$$Y(x_k) = Y_k \qquad (3.29b)$$

$$Y(x) = \sum_{r=0}^{n} a_r T_r\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \qquad (3.30)$$

We shall collocate (3.29) at the only point x_{k+n}, and interpolate (3.30) at the n-points x_{k+r}, $r = 0(1)n - 1$ and thus have

$$\left.\begin{aligned} Y'(x_{k+n}) &= f_{k+n} \\ Y(x_{k+r}) &= Y_{k+r}\ , \quad r = 0(1)n-1 \end{aligned}\right\} \qquad (3.31)$$

Let us now consider specific cases of (3.30) – (3.31).

3.4.1 A One Step Method

When $n = 1$ in (3.30) – (3.31) we have that

$$Y'(x_{k+1}) = f_{k+1}$$

$$Y(x_k) = Y_k$$

That is

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} Y_k \\ hf_{k+1} \end{pmatrix}$$

whose solution yields the values

$$a_0 = Y_k + \frac{1}{2}hf_{k+1}$$

$$a_1 = \frac{1}{2}hf_{k+1}$$

We insert these in (3.30) for $n = 1$ to have the continuous scheme

$$Y(x) = \alpha_0(x)Y_k + h\beta_1(x) f_{k+1} \qquad (3.32)$$

where $\alpha_0(x) = 1, \quad \beta_1(x) = \dfrac{(x - x_k)}{h}$

At the grid point x_{k+1} this yields the one-step backward differentiation method.

$$Y_{k+1} - Y_k = hf_{k+1} \tag{3.33}$$

From this we obtain f_{k+1} for the proposed continuous scheme (3.32).

3.4.2 A Two-Step Method

For a two-step formulation, when $n = 2$ in (3.30) – (3.31) we have

$Y(x_k) = Y_k,, Y(x_{k+1}) = Y_{k+1}$

$Y'(x_{k+2}) = f_{k+2}$

This yields system

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Y_k \\ Y_{k+1} \\ hf_{k+2} \end{pmatrix}$$

whose solution gives the values

$$a_0 = \frac{1}{6}\left(5Y_{k+1} + Y_k + hf_{k+2}\right)$$

$$a_1 = \frac{1}{3}\left(2Y_{k+1} - 2Y_k + hf_{k+2}\right)$$

$$a_2 = \frac{1}{6}\left(Y_k - Y_{k+1} + hf_{k+2}\right)$$

By inserting these in (3.30) with $n = 2$ we get the continuous scheme

$$Y(x) = \alpha_0(x)Y_k + \alpha_1(x)Y_{k+1} + h\beta_2(x)\, f_{k+2} \tag{3.34}$$

where

$$\alpha_0(x) = \frac{(x - x_k)^2}{3h^2} - \frac{(x - x_k)}{h} + 2$$

$$\beta_1(x) = -\frac{(x - x_k)^2}{3h^2} + \frac{(x - x_k)}{h} - 1$$

$$\beta_2(x) = \frac{2(x - x_k)^2}{3h^2} - \frac{(x - x_k)}{h}$$

The corresponding discrete form is the two-step backward differentiation formulae

$$Y_{k+2} - \frac{4}{3}Y_{k+1} + \frac{1}{3}Y_k = \frac{2}{3}hf_{k+2} \tag{3.35}$$

3.4.3 A Three-Step Method

For the case when $n = 3$ in (3.30) – (3.31) we have

$Y(x_k) = Y_k ,, Y(x_{k+1}) = Y_{k+1},$

$Y(x_{k+2}) = Y_{k+2},, Y'(x_{k+3}) = f_{k+3}$

From this we get

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 27 & -9 & -21 & 23 \\ 27 & 9 & -21 & -23 \\ 0 & 2 & 8 & 18 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} Y_k \\ 27Y_{k+1} \\ 27Y_{k+2} \\ 3hf_{k+3} \end{pmatrix}$$

whose solution yields the values

$$a_0 = \frac{1}{352}\left(91Y_k + 36Y_{k+1} + 225Y_{k+2} + 42 f_{k+3}\right)$$

$$a_1 = \frac{3}{704}\left(-69Y_k - 168Y_{k+1} + 237Y_{k+2} + 46hf_{k+3}\right)$$

$$a_2 = \frac{9}{352}\left(13Y_k - 20Y_{k+1} + 7Y_{k+2} + 6hf_{k+3}\right)$$

$$a_3 = \frac{27}{704}\left(-3Y_k + 8Y_{k+1} - 5Y_{k+2} + 2 f_{k+3}\right)$$

These values inserted in (3.31) for *n = 3* yields the continuous scheme

$$Y(x) = Y_k + h\left[\beta_0(x) f_k + \beta_1(x) f_{k+1} + \beta_2(x) f_{k+2} + \beta_3(x) f_{k+3}\right] \qquad (3.36)$$

where

$$\alpha_0(x) = -\frac{3(x-x_k)^3}{22h^3} - \frac{10(x-x_k)^2}{11h^2} - \frac{-39(x-x_k)}{22h} + 1$$

$$\alpha_1(x) = \frac{4(x-x_k)^2}{11h^2} - \frac{23(x-x_k)}{11h} + 30$$

$$\alpha_2(x) = -\frac{5(x-x_k)^3}{22h^3} + \frac{10(x-x_k)^2}{11h^2} + \frac{13(x-x_k)}{11h} + 1$$

$$\beta_3(x) = \frac{(x-x_k)^3}{11h^3} - \frac{3(x-x_k)^2}{11h^2} + \frac{2}{11}$$

The corresponding discrete initial value solver is the three-step backward differentiation formulae

$$Y_{k+3} - \frac{8}{11}Y_{k+2} + \frac{9}{11}Y_{k+1} - \frac{2}{11}Y_k = \frac{6}{11}hf_{k+3} \qquad (3.37)$$

We determine f_{k+3} from (3.37) for the continuous scheme (3.36). We proceed in this manner to obtain the results in the Table 1 below which represents values for the k-step method

$$\sum_{r=0}^{k} \alpha_r Y_{n+r} = h\beta k f_{n+k}, \quad k = 1(1)6$$

Table 1:Coefficients of Backward Differentiation Methods

K	α_0	α_1	α_2	α_3	α_4	α_5	α_6	β_k
1	-1	1						1
2	$\frac{1}{3}$	$\frac{-4}{3}$	1					$\frac{2}{3}$
3	$\frac{-2}{11}$	$\frac{9}{11}$	$\frac{-18}{11}$	1				$\frac{6}{11}$
4	$\frac{3}{25}$	$\frac{-16}{25}$	$\frac{36}{25}$	$\frac{48}{25}$	1			$\frac{12}{25}$
5	$\frac{-12}{137}$	$\frac{75}{137}$	$\frac{-200}{137}$	$\frac{300}{137}$	$\frac{-300}{137}$	1		$\frac{60}{137}$
6	$\frac{10}{147}$	$\frac{-72}{147}$	$\frac{225}{147}$	$\frac{-400}{147}$	$\frac{450}{147}$	$\frac{-360}{147}$	1	$\frac{60}{147}$

4..0 Accuracy of the new continuous scheme

We state here with proof a theorem in support of the accuracy of the Chebyshev based methods viz-a-viz the power series methods.

Theorem **4.1**

Suppose

(i) *y(x)* is continuous in the closed interval *[a,b]*

(ii) y(x) is a solution on *[o,b]* of the initial value problem (1.1)

(iii) $Y_1(x) = \sum_{r=0}^{n} c_r T_r(x) \cong y(x) \;\; on \;\; [o,b]$

(iv) $Y_2(x) = \sum_{r=0}^{n} d_r x_r \cong y(x) \;\; on \;\; [o,b]$

Then

$Y_1(x)$ is a better approximant of *y(x)* on *[o,b]* than $Y_2(x)$.

Proof

Let us define the error function.

$$e_1(x) = y(x) - Y_1(x) \equiv \sum_{r=0}^{\infty} \bar{c}_r T_r(x) - \sum_{r=0}^{n} c_r T_r(x)$$

$$e_2(x) = y(x) - Y_2(x) \equiv \sum_{r=0}^{\infty} \bar{d}_r x^r - \sum_{r=0}^{n} d_r x^r$$

Now we need to show that

$$|e_1(x)| \leqq |e_2(x)| \;\forall\, x \varepsilon [o,b]$$

Now

$$|e_1(x)| = |y(x) - Y_1(x)|$$

$$= \left| \sum_{r=0}^{\infty} \bar{c}_r T_r(x) - \sum_{r=0}^{n} c_r T_r(x) \right|$$

$$= \left| \sum_{r=0}^{n} (\bar{c}_r - c_r) T_r(x) + \sum_{r=n+1}^{\infty} \bar{c}_r T_r(x) \right|$$

$$\leq \left| \sum_{r=0}^{n} (\bar{c}_r - c_r) T_r(x) \right| + \left| \sum_{r=n+1}^{\infty} \bar{c}_r T_r(x) \right|$$

$$\leqq \sum_{r=0}^{n} \left| \bar{c}_r - c_r \right| + \sum_{r=n+1}^{\infty} \left| \bar{c}_r \right|$$

as $n \to \infty$, we have that

$$\left| e_1(x) \right| \leq \sum_{r=n+1}^{\infty} \left| \bar{c}_r \right|$$

also

$$\left| e_2(x) \right| = \left| y(x) - Y_2(x) \right|$$

$$= \sum_{r=0}^{\infty} \bar{d}_r x^r - \sum_{r=0}^{n} d_r x^r$$

$$\left| e_2(x) \right| = \left| \sum_{r=0}^{n} \left(\bar{d}_r - d_r \right) x^r + \sum_{n=1}^{\infty} \bar{d}_r x^r \right|$$

$$\leq \sum_{r=0}^{n} \left| \bar{d}_r - d_r \right| \left| b^r \right| + \sum_{r=n+1}^{\infty} \left| \bar{d}_r \right| \left| b^r \right|$$

as $n \to \infty$ we expect that $c_r = \bar{c}_r = d_r = \bar{d}_r$ Hence, $\left| e_2(x) \right| \leqq \sum_{n+1}^{\infty} \left| \bar{d}_r \right| \left| b^r \right| = \sum_{r=n+1}^{\infty} \left| \bar{c}_r \right| \left| b^r \right|$

***Corollary* 4.2**

Suppose that

(i) *$y(x)$ is a continuous in the closed interval $[x_k, x_{k+n}]$*

(ii) *$y(x)$ is a solution on $[x_k, x_{k+n}]$ of the IVP*

$y(x) = f(x, y(x))$, $\quad y(x_k) = y_k$, $\quad x_k \leq x \leq x_{k+n}$

(iii)

$$Y_{1,k}(x) = \sum_{r=0}^{n} c_r T_r \left(\frac{2x}{nh} - \frac{2k}{n} - 1 \right), \quad x_k \leq x \leq x_{k+n}$$

$$Y_{2,k}(x) = \sum_{r=0}^{n+1} d_r \left(\frac{2x}{nh} - \frac{2k}{n} - 1 \right), \quad x_k \leq x \leq x_{k+n}$$

Then $Y_{1,k}(x)$ approximates $y(x)$ better than does $Y_{2,k}$ *on* $[x_k, x_{k+n}]$.

Proof

Set $c_r = c_{k,r}$, $d_{k,r} = d_r$, $= \bar{c}_r = \bar{c}_{k,r}$

$\bar{d}_r = \bar{d}_{k,r}$, $o = x_k$, $b = x_{k+n}$, $Y_1(x) = Y_{1,k}(x)$ and $Y_2(x) = Y_{2,k}(x)$ in the above proof and the conclusion follows immediately.

5.0 Conclusion

A method which employs the Chebyshev polynomials as basis functions in a multistep collocation technique for a continuous formulation of some classical initial value solvers has been presented. Four popular multistep methods namely the optimal order methods, the Adams-Bashforth explicit methods, the Adams-Moulton implicit methods and the Backward differentiation methods have thus been recovered.

A *(k+1)*–step Adams Bashforth step may be coupled together with a k-step Adams-Moulton method in a predictor-corrector algorithm. The optimal order methods are the most accurate amongst methods of the same step-number, and the backward differentiation formulae are desirable in solution of stiff-problems.

Our major attraction of all these continuous formulation is in their ability to yield solution at the off-points without requiring additional interpolation and at no extra cost (see [2] and [3]).

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