Journal of the Nigerian Association of Mathematical Physics
Volume 10 (November 2006), **249 -256**
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Hydrodynamic dispersion of a reactive solute in Electro-Osmotic flow using quadratic polynomials

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Abstract

The objective of this paper is to study the effect of the dispersion coefficients and the reaction parameter on the hydrodynamic dispersion of a reactive solute in electro-osmotic flow through the method of finite elements using quadratic Lagrange polynomials.

1.0 Introduction

The flow and spread of fluids through porous media such as soils, bead packings, ceramics and concrete plays an important role in a wide variety of environmental and technological processes. Examples include the spreading and clean up of underground hazardous wastes, oil recovery separation processes such as chromatography and catalysis, and the degradation of building materials.

For example chromatographic separation using electric fields to drive electro-osmotic flow are usually performed in packed columns. The role of the packing is to provide a large surface area for solute adsorption and thereby to improve column performance. However, recent advances in manufacturing methods now enable the fabrication of electrochromatographic columns having characteristics transverse dimensions in the micron to submicron range.

Axial dispersion is important in chromatographic process because it tends to spread the solute peaks. As a result, closely packed peaks cannot be resolved when dispersion is excessive. Estimating the magnitude of the dispersion and identifying the conditions leading to minimum dispersion are thus important to optimizing the process.

As a solute is converted in an open column, transverse variations in the velocity field produce transverse variations in the solute concentration. At the same time, transverse diffusion tends to reduce induced concentration gradients. At sufficiently late times transport in the axial direction is just balanced by diffusive transport in the transverse direction. This is the phenomenon of hydrodynamic dispersion. Such dispersion yields a mean axial profile of the solute concentration that is consistent with diffusive transport alone, although the apparent diffusivity is larger than the actual value.

While the study of miscible flow in porous media has been of considerable research, it is difficult to obtain exact solutions of the Navier – stokes and hydrodynamic dispersion equations for the case of flow in porous media. However advances in computer have made it numerically possible to simulate fluid flow in complex geometrics. Many factors which control the invasion of fluids such as viscosity, surface tension forces, the structure of the porous medium, and the external driving force which displaces the fluids can be directly incorporated into computation of fluid dynamics.

2.0 Governing Equation

In this research work we consider the flow of a two dimensional planar transport of a reactive solute in an electro-osmotic flow. The flow is assumed to be incompressible and transport properties are assumed constant. Under this restriction the time dependent concentration field is governed by

$$\frac{\partial c}{\partial t} = Dx\frac{\partial^2 c}{\partial x^2} + Dy\frac{\partial^2 c}{\partial y^2} - U\frac{\partial c}{\partial x} + \lambda c \qquad (2.1)$$

Where c is the local solute concentration t is the time, and u is the local fluid velocity. Dx, Dy are the coefficients of hydrodynamic dispersion in the x and y direction respectively, and λ is the rate of chemical reaction.

Further assuming that flow is steady and that inertial effects are small, the momentum equation may be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\ell}{\mu}\frac{\partial \phi}{\partial x} \qquad (2.2$$

Where μ is the viscosity, ℓ is the net local charge density, and ϕ is the local electric potential.

Finally, for a dielectric constant $\in$ that does not vary with position the poison equation governing the electric field is:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{-\ell}{\in} \qquad (2.3)$$

We shall consider the following boundary and initial conditions

$$\begin{aligned}
& t \le 0; x \ge 0; c = 0 \\
& t \ge 0; x = 0; c = c_0 \\
& x = \infty; c = 0 \\
& y \le 0; c = 0 \\
& and \\
& y \succ 0; c = c_0 \\
& \frac{\partial u}{\partial y}(x,0) = 0, u(0,y) = 0 \\
& u(x,b) = 0, u(l,y) = u_1 \\
& \frac{\partial \phi}{\partial y}(x,0) = 0, \phi(0,y) = 0
\end{aligned} \qquad (2.4)$$

and the local charge density may be related to the electric field potential through the Boltzmann distribution given by: $\ell = -2FZCeSinh(ZF\phi/RT)$

where F is the Faraday constant, Z is the ion charge number, Ce is the bulk fluid ion concentration R is the universal gas constant and T is the temperature. Various forms of (2.1) have been solved by various researchers for various boundary conditions. For example Gardner et al, (1964) Fried (1975) solved the one dimensional case for Zero reaction using the method of characteristics Bear (1979) provides some analytical solutions for the one dimensional problem using the Laplace transform. Batu (1989) provides a generalized analytical solution for the two dimensional case using Laplace transform. Also Celia (1989) solved the one dimensional case with Biodegradation. Dillon (1989) also consider an analytical model of contaminant transport from diffuse sources in saturated porous Media analytically, Robert et al (1989) provided an approximate solution for one-dimensional absorption in unsaturated a finite element model for the diffusion convection equation with applications to air pollution. (Hromadka II and Guymon 1982) use Nodal domain integration model of one dimensional advection-diffusion problem. Barker and Soliman (1982) also solve the case of solute transport in fissured aquifer using method of Laplace transform. Tim and Mostaghimi (1989) used the finite element method to solve the one dimensional form of the transport of pesticides and their metabolites in the unsaturated zone, to mention but a few. But the problem of flow of a solute in an electro-osmotic flow has only been treated sparingly only few literatures exist to the knowledge of the author. For example Griffiths and Roberts (2003) have treated the problem of the case of a non-reactive neutral solute using the method of asymptotic series solution and they only considered the steady state one dimensional problem.

The difference between the above literature and ours is that we are considering a two dimensional problem of the flow of a reactive solute in electro osmotic flow which involves solving three partial differential equations simultaneously.

3.0 Solution of the two dimensional problems using the quadratic Lagranges interpolation functions

We shall proceed to approximate the contaminant dispersion problem by the quadratic LaGrange polynomial as follows for triangular elements.

$$L_i = \frac{1}{2\Delta}\left[a_i + b_i x + c_i y\right]$$

$$L_j = \frac{1}{2\Delta}\left[a_j + b_j x + c_j y\right]$$

$$L_k = \frac{1}{2\Delta}\left[a_k + b_k x + c_k y\right]$$

such that:

$$N_i = 2L_i^2 - L_i, \quad N_L = 4L_i L_j$$

$$N_j = 2L_j^2 - L_j, \quad N_m = 4L_j L_k$$

$$N_k = 2L_k^2 - L_k, \quad N_n = 4L_k L_i$$

so that:

$$C^{(e)} = N_i C_i + N_L C_L + N_j C_j + N_m C_m + N_k C_k + N_n C_n$$

which can be written as $C = N^{(e)} C^{(e)}$ (3.1)

Given that $$\frac{\partial c}{\partial t} = Dx\frac{\partial^2 c}{\partial x^2} + Dy\frac{\partial^2 c}{\partial y^2} - U\frac{\partial c}{\partial x} - \lambda c \qquad (3.2)$$

Putting (3.1) in to (3.2) we have

$$\frac{\partial}{\partial t}\left[N^{(e)}C^{(e)}\right] = Dx\frac{\partial^2}{\partial x^2}\left[N^{(e)}C^{(e)}\right] + Dy\frac{\partial^2}{\partial x^2}\left[N^{(e)}C^{(e)}\right] - Un\frac{\partial}{\partial x}\left[N^{(e)}C^{(e)}\right] - \lambda\left[N^{(e)}C^{(e)}\right] \qquad (3.3)$$

Using the Galerkin criterion we have

$$\iint N^{(e)T}\frac{\partial}{\partial t}\left[N^{(e)}C^{(e)}\right]dxdy = Dx\iint N^{(e)T}\frac{\partial^2}{\partial x^2}\left[N^{(e)}C^{(e)}\right]dxdy + Dy\iint N^{(e)T}\frac{\partial^2}{\partial y^2}\left[N^{(e)}C^{(e)}\right]dxdy$$

$$-Un\iint N^{(e)T}\frac{\partial}{\partial x}\left[N^{(e)}C^{(e)}\right]dxdy - \lambda\iint N^{(e)T}N^{(e)}C^{(e)}dxdy \qquad (3.4)$$

This on simplification gives

$$\frac{\partial C^{(e)}}{\partial t}\iint N^{(e)T}N^{(e)}dxdy = -Dx\iint\frac{\partial N^{(e)T}}{\partial x}.\frac{\partial N^{(e)}}{\partial x}.C^{(e)}dxdy - Dy\iint\frac{\partial N^{(e)T}}{\partial y}.\frac{\partial N^{(e)}}{\partial y}.C^{(e)}dxdy$$

Giving in finite difference form we have:

$$= -\frac{Dx}{3h^2}\left[C_{m-1,n} + 6C_{m,n} + C_{m+1,n} - 2C_{m-\frac{1}{2},n} - 4C_{m+\frac{1}{2},n}\right] - \frac{Dy}{3k^2}\left[3C_{m-1,n} + 2C_{m,n+1} - 8C_{m,n+\frac{1}{2}}\right]$$

$$\frac{1}{180}\left[-\dot{C}_{m-1,n} + 18\dot{C}_{m,n} - \dot{C}_{m+1,n}\ 2\dot{C}_{m-1,n+1} - 2\dot{C}_{m,n+1} - 4\dot{C}_{m-1,n+\frac{1}{2}} - 4\dot{C}_{m+\frac{1}{2},n+\frac{1}{2}} - 4\dot{C}_{m-\frac{1}{2},n+1}\right]$$

$$-\frac{U}{15h}\left[\begin{matrix} C_{m-1,n} + 4C_{m,n} + C_{m+1,n} - C_{m-1,n+1} + C_{m,n+1} + 3C_{m-\frac{1}{2},n} - 3C_{m+\frac{1}{2},n} - C_{m-1,n+\frac{1}{2}} - 3C_{m-\frac{1}{2},n+\frac{1}{2}} \\ -3C_{m,n+\frac{1}{2}} - C_{m+\frac{1}{2},n+\frac{1}{2}} \end{matrix}\right]$$

$$-\frac{\lambda}{180}\left[-C_{m-1,n}+18C_{m,n}-C_{m+1,n}-2C_{m-1,n+1}-2C_{m,n+1}-4C_{m-1,n+\frac{1}{2}}-4C_{m+\frac{1}{2},n+\frac{1}{2}}-4C_{m-\frac{1}{2},n+1}\right]$$

(3.5)

which on collecting like terms we have

$$\frac{1}{180}\left[-\dot{C}_{m-1,n}+18\dot{C}_{m,n}-\dot{C}_{m+1,n}-2\dot{C}_{m-1,n+1}-2\dot{C}_{m,n+1}-4\dot{C}_{m-1,n+\frac{1}{2}}-4\dot{C}_{m+\frac{1}{2},n+\frac{1}{2}}-4\dot{C}_{m-\frac{1}{2},n+1}\right]$$

$$=\left[-\frac{Dx}{3h^2}-\frac{3Dy}{3k^2}-\frac{U_n}{15h}+\frac{\lambda}{180}\right]C_{m-1,n}+\left[-\frac{2Dx}{h^2}+\frac{4U_n}{15h}-\frac{\lambda}{10}\right]C_{m,n}+\left[-\frac{Dx}{3h^2}-\frac{2Dy}{3k^2}-\frac{U_n}{15h}+\frac{\lambda}{180}\right]C_{m+1,n}$$

$$+\left[2\frac{Dx}{3h^2}--\frac{U_n}{15h}\right]C_{m-\frac{1}{2},n}+\left[\frac{4Dx}{3h^2}+\frac{U_n}{15h}\right]C_{m+\frac{1}{2},n}+\left[8\frac{Dy}{3k^2}+\frac{U_n}{15h}\right]C_{m,n+\frac{1}{2}}+\left[\frac{U_n}{15h}+\frac{\lambda}{90}\right]C_{m-1,n+1}$$

$$+\left[-\frac{U_n}{15h}+\frac{\lambda}{90}\right]C_{m,n-1}+\left[-\frac{U_n}{15h}+\frac{\lambda}{45}\right]C_{m-1,n+\frac{1}{2}}+\left[\frac{U_n}{15h}+\frac{\lambda}{45}\right]C_{m+\frac{1}{2},n+\frac{1}{2}}-C_{m-\frac{1}{2},n+\frac{1}{2}}-4C_{m-\frac{1}{2},n+\frac{1}{2}}$$

We shall now proceed to apply the finite difference scheme to the time variable and the trapezoidal rule to obtain the cranknicolson of grid scheme. Taking

$$A=90\left[-\frac{Dx}{3h^2}-\frac{Dy}{k^2}-\frac{U_n}{15h}+\frac{\lambda}{180}\right]\Delta t,\quad B=90\left[-\frac{2Dx}{h^2}+\frac{4U_n}{15h}-\frac{\lambda}{10}\right]\Delta t$$

$$I=90\left[-\frac{Dx}{3h^2}-\frac{2Dy}{3k^2}-\frac{U_n}{15h}+\frac{\lambda}{180}\right]\Delta t,\quad D=90\left[-\frac{2Dx}{3h^2}-\frac{U_n}{15h}\right]\Delta t$$

$$E=90\left[\frac{4Dx}{3h^2}+\frac{U_n}{15h}\right]\Delta t,\quad F=90\left[\frac{8Dy}{3k^2}+\frac{U_n}{15h}\right]\Delta t,\quad G=90\left[\frac{U_n}{15h}+\frac{\lambda}{90}\right]\Delta t,$$

$H=90\left[\frac{U_n}{15h}+\frac{\lambda}{90}\right]\Delta t,\quad J=90\left[\frac{U_n}{15h}+\frac{\lambda}{45}\right]\Delta t$. We obtain:

$$[-1-A]C^{r+1}_{m-1,n}+[+18-B]C^{r+1}_{m,n}+[-1-I]C^{r+1}_{m+1,n}+[-2-H]C^{r+1}_{m,n+1}+[-4-J]C^{r+1}_{m-1,n+\frac{1}{2}}+[-4-2\Delta t]C^{r+1}_{m-\frac{1}{2},n+1}+[-2-G]C^{r+1}_{m-1,n+1}-4C^{r+1}_{m+\frac{1}{2},n+\frac{1}{2}}$$

$$-\Delta C^{r+1}_{m-\frac{1}{2},n}-EC^{r+1}_{m+\frac{1}{2},n}-FC^{r+1}_{m,n+\frac{1}{2}}=[A-1]C^{r}_{m-1,n}+[B+18]C^{r}_{m,n}+[I-1]C^{r}_{m+1,n}+[G-2]C^{r}_{m-1,n+1}+[H-2]C^{r}_{m,n+1}+[J+4]C^{r}_{m-1,n+\frac{1}{2}}$$

$$+[4-2\Delta t]C^{r}_{m-\frac{1}{2},n+1}-4C^{r}_{m+\frac{1}{2},n+\frac{1}{2}}+EC^{r}_{m+\frac{1}{2},n}+FC^{r}_{m,n+\frac{1}{2}}-\frac{3}{2}\Delta tC^{r}_{m-\frac{1}{2},n+\frac{1}{2}}+DC^{r}_{m-\frac{1}{2},n} \qquad (3.5)$$

(3.5)) is the iterative formula for the quadratic Lagrange interpolation functions for the concentration.

4.0 Solution of the two dimensional momentum equation using the quadratic Lagrange's interpolation functions

We shall in this section proceed to formulate the finite elements solution of the two dimensional problem using the quadratic interpolation function for triangular elements.

Recalling that the two dimensional momentum equation to be given by:

$$\frac{\partial^2 u}{dx^2}+\frac{\partial^2 u}{dy^2}=\frac{e}{\mu}\frac{\partial\phi}{\partial x} \qquad (4.1)$$

together with the boundary conditions

The function is as given in the last section. We shall proceed to use the Galerkin's method as usual. Let:

$$U^{(e)}=\sum_i N_i^{(e)}U_i \qquad (4.2)$$

We have:

$$\frac{\partial^2\left[N^{(e)}U^{(e)}\right]}{dx^2}+\frac{\partial^2\left[N^{(e)}U^{(e)}\right]}{dy^2}=\frac{e}{\mu}\frac{\partial\left[N^{(e)}\phi^{(e)}\right]}{\partial x} \qquad (4.3)$$

So that we shall have:

$$\iint N^{(e)T}\frac{\partial^2\left[N^{(e)}U^{(e)}\right]}{dx^2}dxdy+\iint N^{(e)T}\frac{\partial^2 N^{(e)}U^{(e)}}{dx^2}dxdy=\frac{e}{\mu}\iint N^{(e)T}\frac{\partial\left[N^{(e)}\phi^{(e)}\right]}{\partial x}dxdy$$

Which on integration we have:

$$\int N^{(e)T}\frac{\partial N^{(e)}U^{(e)}}{dx}dy-\iint\frac{\partial N^{(e)T}}{dx}.\frac{\partial N^{(e)}U^{(e)}}{dy}dxdy+\int N^{(e)T}\frac{\partial N^{(e)}U^{(e)}}{dy}dy-\iint\frac{\partial N^{(e)T}}{dy}.\frac{\partial N^{(e)}U^{(e)}}{dy}dxdy$$

$$=\frac{e}{\mu}\iint N^{(e)}\frac{\partial N^{(e)}\phi^{(e)}}{dy}dxdy$$

this simplifies to give:

$$-\iint\frac{\partial N^{(e)T}}{dx}.\frac{\partial N^{(e)}U^{(e)}}{dx}dxdy-\iint\frac{\partial N^{(e)T}}{dy}.\frac{\partial N^{(e)}U^{(e)}}{dy}dxdy=\frac{e}{\mu}\iint N^{(e)T}\frac{\partial N^{(e)}\phi^{(e)}}{dx}dxdy$$

so that:

$$\iint\frac{\partial N^{(e)T}}{dx}.\frac{\partial N^{(e)}U^{(e)}}{dx}dxdy+\iint\frac{\partial N^{(e)T}}{dy}.\frac{\partial N^{(e)}U^{(e)}}{dy}dxdy=-\frac{e}{\mu}\iint N^{(e)T}\frac{\partial N^{(e)}\phi^{(e)}}{dx}dxdy$$

In finite difference form

$$\frac{k^2}{12\Delta}\left[U_{m-1,n}+6U_{m,n}+U_{m+1,n}-4U_{m-\frac{1}{2},n}-4U_{m+\frac{1}{2},n}\right]+\frac{h^2}{12\Delta}\left[3U_{m-1,n}+2U_{m,n+1}-4U_{m,n+\frac{1}{2}}\right]$$

$$=\frac{-ek}{30\mu}\left[\phi_{m-1,n}+4\phi_{m,n}+\phi_{m+1,n}-\phi_{m-1,n+1}+\phi_{m,n+1}+3\phi_{m-\frac{1}{2},n}-3\phi_{m+\frac{1}{2},n}-\phi_{m-1,n+\frac{1}{2}}+3\phi_{m+\frac{1}{2},n+\frac{1}{2}}-3\phi_{m,n+\frac{1}{2}}+\phi_{m+\frac{1}{2},n+\frac{1}{2}}\right]$$

Since $\Delta = hk$ we multiply through by $12\,hk$ we have

$$k^2\left[U_{m-1,n}+6U_{m,n}+U_{m+1,n}-4U_{m-\frac{1}{2},n}-4U_{m+\frac{1}{2},n}\right]+h^2\left[3U_{m-1,n}+2U_{m,n+1}-8U_{m,n+\frac{1}{2}}\right]$$

$$=\frac{-12ek^2h}{30\mu}\left[\phi_{m-1,n}+4\phi_{m,n}+\phi_{m+1,n}-\phi_{m-1,n+1}+\phi_{m,n+1}+3\phi_{m-\frac{1}{2},n}-3\phi_{m+\frac{1}{2},n}-\phi_{m-1,n+\frac{1}{2}}+3\phi_{m-\frac{1}{2},n+\frac{1}{2}}-3\phi_{m,n+\frac{1}{2}}+\phi_{m+\frac{1}{2},n+\frac{1}{2}}\right] \quad (4.4)$$

This is the difference scheme for the momentum equation

3.0 Solution of the two dimensional Electric Field potential using quadratic Lagranges interpolation function

We recall the equation for the two dimensional electric field potential to be together with the boundary conditions

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-\frac{e}{\in} \quad (5.1)$$

Given that

$$\phi^{(e)}=\sum_i N_i^{(e)}\phi_i^{(e)} \quad (5.2)$$

By putting (5.2) into (5.1) and integrating we have

$$\iint N^{(e)}\partial^2\frac{\left[N^{(e)}\phi^{(e)}\right]}{dx^2}dxdy+\iint N^{(e)}\partial^2\frac{\left[N^{(e)}\phi^{(e)}\right]}{dy^2}dxdy=\frac{-e}{\in}\iint N^{(e)T}dxdy \quad (5.3)$$

This simplifies to give $$-\iint\frac{\partial N^{(e)T}}{dx}.\frac{\partial N^{(e)}}{dx}\phi^{(e)}dxdy-\iint\frac{\partial N^{(e)T}}{dy}.\frac{\partial N^{(e)}}{dy}\phi^{(e)}dxdy-\frac{\ell}{\in}\iint N^{(e)T}dxdy$$

carrying out a similar analysis for the electric potential we have

$$\frac{k^2}{12\Delta}\left[\phi_{m-1,n}+6\phi_{m,n}+\phi_{m+1,n}-4\phi_{m-\frac{1}{2},n}-4\phi_{m+\frac{1}{2},n}\right]+\frac{h^2}{12\Delta}\left[3\phi_{m-1,n}+2\phi_{m,n+1}-8\phi_{m,n+\frac{1}{2}}\right]=0$$

Since $\Delta = hk$ we multiply through by $12\,hk$ to give

$$k^2\left[\phi_{m-1,n}+6\phi_{m,n}+\phi_{m+1,n}-4\phi_{m-\frac{1}{2},n}-\phi_{m+\frac{1}{2},n}\right]+h^2\left[3\phi_{m-1,n}+2\phi_{m,n+1}-8\phi_{m,n+\frac{1}{2}}\right]=0$$

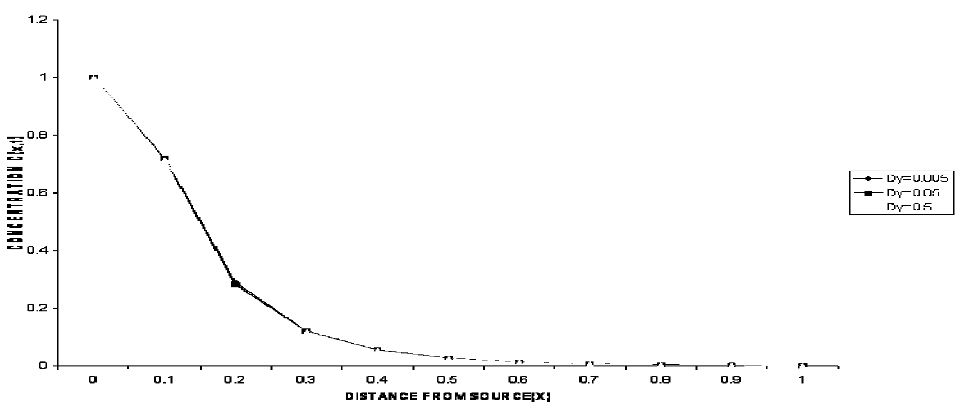
This gives

$$\left[k^2 + 3h^2\right]\phi_{m-1,n} + \left[k^2 + 2h^2\right]\phi_{m,n+1} + k^2\left[6\phi_{m,n} - 4\phi_{m-\frac{1}{2},n} - 4\phi_{m+\frac{1}{2},n}\right] - 8h^2\phi_{m,n+\frac{1}{2}} = 0$$

4.0 Discussion

Figure 1 ows the impact of changes in the vertical dispersion for quadratic polynomials for Dx = 0.1, λ =-1000, Un = 0.133, h = 0.1, k = 0.5, dt = 0.5 it is observed that as the dispersion coefficient increases the concentration profile decreases. Figure .2 we show the impact of variation in the longitudinal dispersion coefficient for Dx = 0.005, λ =-1000 and Un=0.0133, h-0.1, k = 0.5, dt=0.5. It is also observed that the concentration profile decrease with increase in the dispersion coefficient. That is an increase in the dispersion process reduces the concentration of the contaminant for both the longitudinal and vertical dispersion coefficients

FIG.1 EFFECT OF VARIATION IN VERTICAL DISPERSION COEFFICIENT



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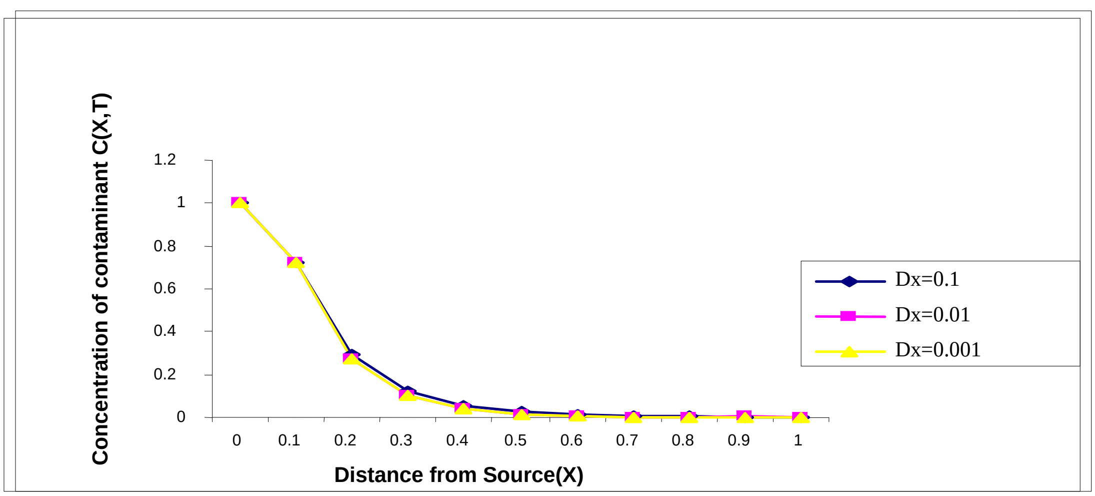

Figure 2: Effect of vriation longitudinal dispersion coefficient for quadratic polynomials

Lastly Figure 3 shows the impact of the reaction parameter for Dx = 0.1, Dy = 0.5 and Un = 0.5, h=0.1, k = 0.5, dt = 0.5.It is observed that increase in the the reaction parameter decrease the value of the concentration. Meaning that when the rate of reaction is high the overall concentration of the initial contaminant decreases.

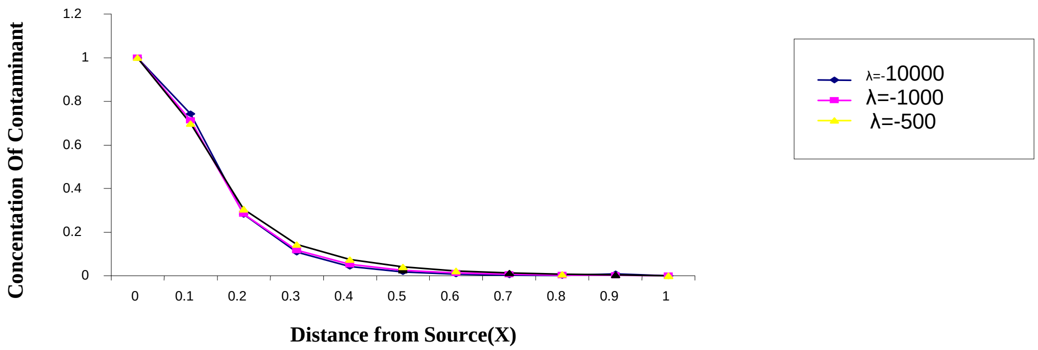

Figure 3: Effect of Variation in Reaction Parameter

In the above cases it can be observed that the concentration profile decreases steadily in the axial direction until a steady state is reached.

5.0 Summary

We have been able to provide a finite element solution to the problem of hydrodynamic dispersion of a reactive solute in electro-osmotic flow using quadratic triangular element for the two dimensional problem. Numerical oscillation has been successfully minimized. This shows the efficiency of the method in simulating hydrodynamic dispersion of reactive solute. Also in this work we have been able to numerically show the impact of the reaction parameter in a two dimensional electro-osmotic flow which does not exist in literature.

Reference

[1] Baker A.J; Soliman M.O (1982); Analysis of Finite Element Algorithm for Numerical Predictions in Water Resources Research J. adv. Water Resources.

[2] Batu V. (1989) A Generalized Two-Dimensional Analytical Solution of Hydrodynamic Dispersion in Bounded Media with the First Type Boundary Condition at the Source. Water Resources Research vol. 25.

[3] Bear J. (1979) Hydraulics of groundwater McGraw hill. New York

[4] Bird R.B, Warren E. S. Lightfoot E.N (2002) Transport Phenomena, John Wiley & Sons Inc. New York.

[5] Celia M.A and Gray W.G (1992) Numerical Methods for Differential Equations Prentice Hall Int. London.

[6] Dillon P.J (1989) An Analytical Model of Contaminant Transport from Diffuse Sources in Saturated Porous Media, Water Resources Research.

[7] Ding J.Y. and Wu S.C (1994) Lab. Studies of the effects of Dissolved Organic Material on the Adsorption of Organochlorine Pesticides by Sediments and Transport in Rivers: Water Science Technology Vol 28.

[8] Gerald C.F; Wheatly P.O (1999) Applied Numerical Analysis 6th Edition, Pearson Education Inc.

[9] Griffiths S.K; Roberts R.H (2001) Hydrodynamic Dispersion of Neutral Non-Reactive Solute in Electroosmotic Flow: Tech. Proc. 2001 Conf. of Modeling and Simulation of Microsystems.

[10] Hromadka. T.V. and Guymon G.L (1982) Nodal Domain Integration Model of One-Dimensional Advection – Diffusion: J. Adv Water Resources.

[11] Liu K, Enfield C.G. and Mravik S.C (1991) Evaluation of Sorption Models in the Simulation of Naphthalene Transport through Saturated Soils Groundwater.

[12] Ostfeld A, Muzaifair E. and Lansey K.E (1999) Analytical Groundwater Flow Solutions for Channel-Aquifer Interaction (J. of Irrigation and Drainage Engineering).

[13] Qiao R. and Aluru N.R (2002): Transient Analysis of Electroosmotic Flow in Nano-Diameter Channels. Technical Proceedings of 2002 Int. Conf. Of Modeling and Simulations of Microsystems.

[14] Tim U.S. and Mostaghimi S. (1989); Model for Predicting Transport of Pesticide and their Metabolites in the Unsaturated Zone. Water Resources Bulletin America Water Resources Association.