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Vorticity determination in a hydraulic jump by application of method of characteristics

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Abstract

The method of characteristics for solving systems of partial differential equations coupled with jump conditions is used in analysing flow downstream of a hydraulic jump instead of the normal analytical approach adopted in Hornung [1]. It is shown that the method of characteristics together with the jump conditions can correctly be used as an alternative method to determine the mean vorticity downstream of the hydraulic jump as a function of the Froude number and height ratio. The mean vorticity does not increase from zero as a function of Froude number minus one but, however, it approaches a constant value at large Froude number. The present work extends the model of Hornung [1] to include non linear velocity profile used in calculating the torque with a view to determining the mean vorticity. The result obtained by this method generalizes that of [1].

1.0 Introduction

Hydraulic jump can be understood as a sudden and turbulent passage of a flowing liquid from a low stage below critical depth to a high stage above critical depth during which the velocity changes from supercritical to subcritical. The jump is an example of non-uniform flow in an open channel and it frequently appears as a stationary phenomenon in the steady flow of a stream (Lighthill [2]). It proves to be useful and instructive for many different purposes to describe flow at large Reynolds number and Froude number primarily in terms of the distribution of vorticity since at large Reynolds number and Froude number the flow downstream of a hydraulic jump will be seen to be rotational (Hornung [1]).

In his paper Hornung [1] determined the mean vorticity downstream of a hydraulic jump as a function of the Froude number by an analytic approach through conservation of angular momentum. Although the total momentum is conserved across the jump, the total mechanical energy dissipates. This energy dissipation is attributed to turbulence generated at the jump. Thus, for instance, Rouse et al [3] modelled a hydraulic jump as a flow expansion and interpreted the turbulence as separation eddies, while Rajaratnam [4] and Narayanam [5] modelled the jump as a wall jet. Other contributors to vortictiy determination include, notably, Yeh [6], Gharib and Wiegand [7], Rood [8], Wu [9], Longuet-Higgins [10], and so on.

In this paper the flow downstream of a hydraulic jump has been analysed and the mean vorticity determined using the method of characteristics coupled with jump conditions. The paper extends the work of Hornung by including the nonlinear velocity profile. The mean vorticity, in dimensionless form, approaches a maximum value of 2 as $F \to \infty$

2.0 Jump Conditions

Here, it is assumed that uniform flow occurs both upstream and downstream of the hydraulic jump and that the resistance of the channel bottom is negligible. We apply the conservation of mass and momentum to a control volume of the hydraulic jump (Figure 1) in order to derive the classical equations connecting the conditions upstream and downstream of a hydraulic jump. (Figure 1).

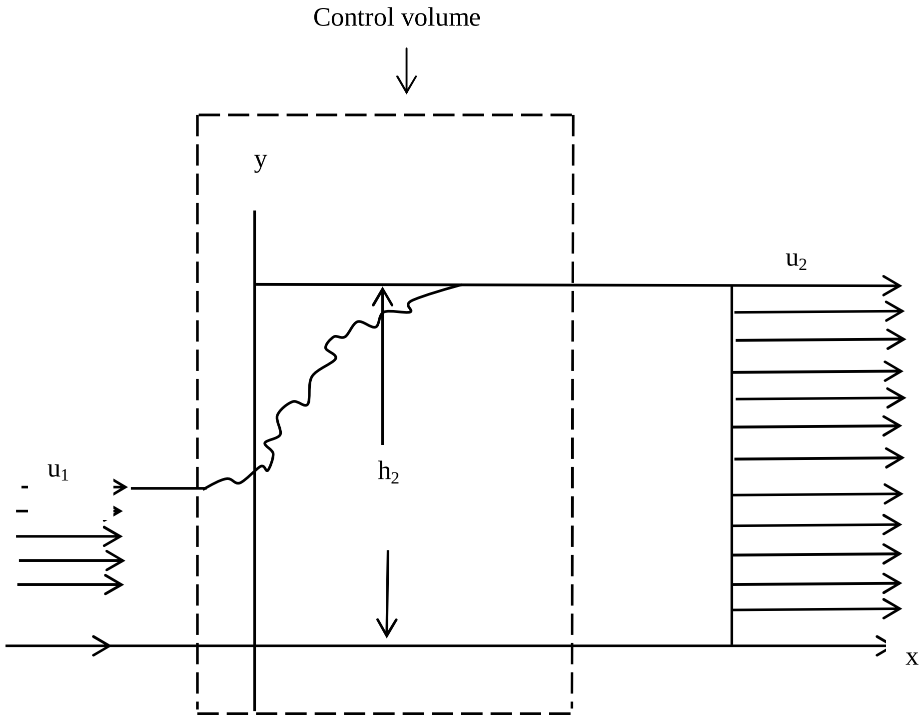

Figure 1: Hydraulic jump with control volume

From the quantities given in figure 1 the continuity equation is given as

$$u_1h_1 = u_2h_2 \qquad (2.1)$$

where u_1,u_2 are the velocities before and after the jump and h_1 and h_2 are the corresponding depths. The momentum equation becomes

$$\tfrac{1}{2}\rho g h_1^2 + \rho u_1^2 h_1 = \tfrac{1}{2}\rho g h_2^2 + \rho u_2^2 h_2 \qquad (2.2)$$

Here the suffixes 1, 2 are used to denote quantities upstream and downstream of the jump respectively
Equation (2.2) can be expressed as

$$h_2^2 - h_1^2 = 2\frac{\left(u_1^2h_1 - u_2^2h_2\right)}{g} \qquad (2.3)$$

Substituting (2.1) into (2.3) and simplifying we obtain

$$\frac{u_1^2}{gh_1} = \frac{1}{2}\frac{h_2^2}{h_1^2}\left(1+\frac{h_1}{h_2}\right) \qquad (2.4)$$

The non - dimensional Froude number F is defined by

$$F = \frac{u_1^2}{gh_1} \qquad (2.5)$$

so that (2.4) can be expressed in the form

$$F = \frac{H^2}{2}\left(1+\frac{1}{H}\right) \qquad (2.6)$$

Here
$$H = \frac{h_2}{h_1} \qquad (2.7)$$

is the height ratio, with limits H ➞ 1 as F ➞ 1. F in (2.5) is the Froude number before the hydraulic jump.

3.0 Conditions across the jump with downstream vorticity

Here the downstream velocity distribution (see [1]) is given by

$$u = u_2 + \omega\left(\frac{h_2}{2} - y\right) \tag{3.1}$$

where u_2 is the downstream mean velocity, ω is the mean vorticity, y is the distance measured vertically upward from the bottom of the channel as in Figure 2.

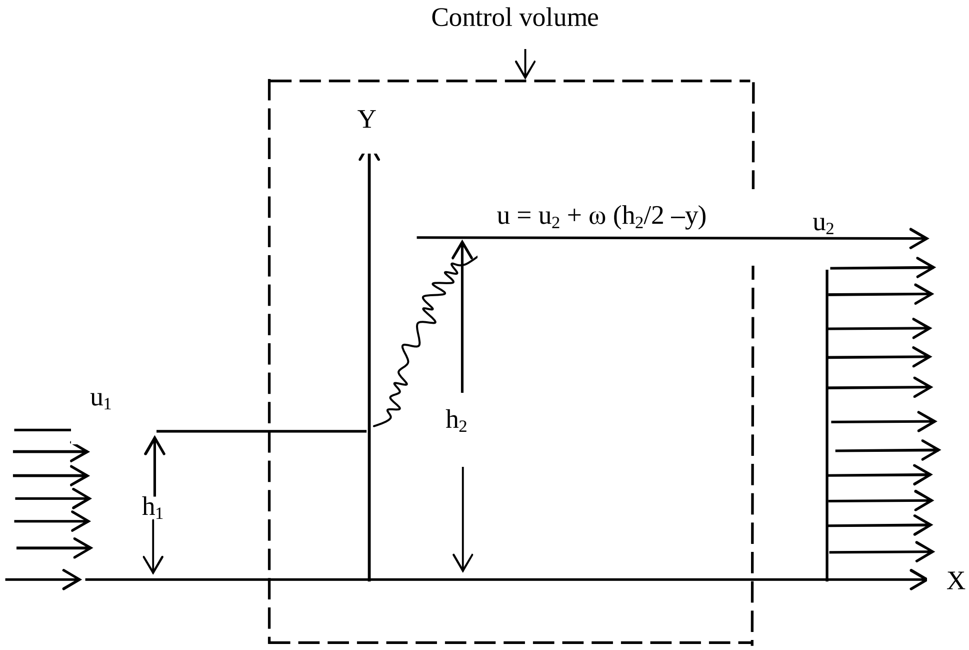

Figure 2: Hydraulic jump with control volume and finite mean vorticity downstream

This change in velocity distribution affects only the momentum balance, which has to be modified. The modified form is

$$\frac{gh_1}{u_1^2} + 2 = \frac{h_2^2 gh_1}{h_1^2 u_1^2} + \frac{2h_1 + \mathrm{I}}{h_2} \tag{3.2}$$

where

$$\mathrm{I} = \frac{2h_1}{h_2}\int_0^1 \left(\frac{2\omega}{u_2}\right)\left(\frac{h_2 - y}{2}\right) + \frac{\omega^2}{u_2^2}\left(\frac{h_2 - y}{2}\right)^2 \Bigg] d\left(\frac{y}{h_2}\right) \tag{3.3}$$

Using the change of variable $\qquad \dfrac{y}{h_2} = t \qquad\qquad (3.4)$

the integral (3.3) becomes

$$\mathrm{I} = \frac{2h_1}{h_2}\int_0^1 \left(\frac{\omega h_2^2}{u_2} - \frac{2\omega t h_2}{u_2} + \frac{\omega^2 h_2^2}{4u_2^2} + \frac{\omega^2 t^2 h_2^2}{u_2^2} - \frac{\omega^2 t h_2^2}{u_2^2}\right) dt \tag{3.5}$$

Integrating (3.5) with respect to t from t = 0 to t = 1 we find $\mathrm{I} = \dfrac{2h_1}{h_2}\left(\dfrac{\omega^2 h_2^2}{4u_2^2} + \dfrac{\omega_2^2 h_2}{3u_2^2} - \dfrac{\omega^2 h_2^2}{2u_2^2}\right)$

which yields after simplification

$$\mathrm{I} = \frac{\omega^2 h_2 h_1}{6u_2^2} \tag{3.6}$$

or $$\mathrm{I} = \frac{\omega^2 h_2^2}{6Hu_2^2} \tag{3.7}$$

(using (2.7)). Equation (3.7) can be written as

$$\mathrm{I} = \frac{1\Omega^2}{6H} \tag{3.8}$$

where $$\Omega = \frac{\omega h_2}{u_2} \tag{3.9}$$

Substituting (3.8) into (3.2) and writing the result in dimensionless variables we obtain

$$\frac{1+2}{F} = H^2 \frac{1}{F} + \frac{2}{H} + \frac{1\Omega}{6H} \tag{3.10}$$

which on re-arrangement yields

$$\frac{1}{F}\left(1 - H^2\right) = \frac{12\left(1 - H\right) + \Omega^2}{6H} \tag{3.11}$$

Multiplying both sides of (3.11) by $\frac{H^2}{2}$ and simplifying we obtain

$$\frac{H^2}{2F}\left(1 + \frac{1}{H}\right) = \frac{1 - \Omega^2}{12\left(H - 1\right)} \tag{3.12}$$

4.0 Method of characteristics and calculation of torque through the use of nonlinear velocity profile

Let t* be the torque per unit lateral distance and introduce the dimensionless torque given by

$$T = \frac{t^*}{\rho u_1^2 h_1^2} \tag{4.1}$$

To determine T we consider the system of equations (the continuity equation and the momentum equation):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1\partial p}{\rho\partial y} - g \tag{4.3}$$

Equation (4.3) can be written as

$$\frac{\partial v}{\partial x} + \frac{v}{u}\frac{\partial v}{\partial y} = -\frac{1\partial p}{\rho u\partial y} - \frac{g}{u} \tag{4.4}$$

In matrix notation the PDES (4.2) and (4.4) become

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \\ \frac{\partial v}{\partial x} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \\ 0 & \frac{v}{u} \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \frac{1}{\rho u}\frac{\partial p}{\partial y} - \frac{g}{u} \end{pmatrix} \tag{4.5}$$

or $$\underline{u}_x + A\underline{u}_y = \underline{d} \tag{4.6}$$

where $A = \begin{pmatrix} 0 & 1 \\ \\ 0 & \frac{v}{u} \end{pmatrix}$, $\underline{u}_x = \begin{pmatrix} u_x \\ \\ v_x \end{pmatrix}$, $\underline{u}_y = \begin{pmatrix} u_y \\ \\ v_y \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 0 \\ \\ \frac{-1}{\rho u}\frac{\partial p}{\partial y} - \frac{g}{u} \end{pmatrix}$

The eigenvalues of the matrix A are $\lambda_1 = 0,\ \lambda_2 = \frac{v}{u}$. Thus the gradients of the characteristic curves are given by

$$\frac{dy}{dx} = 0,\ \frac{dy}{dx} = \frac{v}{u} \tag{4.7}$$

in the *xy*- plane, which trace the progress of the waves. Their integration yields the characteristics of the system of equations (4.2) and (4.3). The integration of the first equation of (4.7) yields the characteristic curve y = constant. The second equation can only be solved if we know the forms of u and v. The eigenvectors corresponding to λ_1 and λ_2 are respectively. $\underline{u}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{v}^{(2)} = \begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix}$

The fundamental matrix becomes $P = \begin{pmatrix} 1 & 1 \\ 0 & \frac{v}{u} \end{pmatrix}$ so that $P^{-1} = \begin{pmatrix} 1 & -\frac{u}{v} \\ 0 & \frac{v}{u} \end{pmatrix}$. Now

$$P^{-1}AP = \Lambda \qquad (4.8)$$

where $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \frac{v}{u} \end{pmatrix}$ = diagonal matrix. Equation (4.8) implies

$$\mathrm{A} = \mathrm{P}\,\Lambda\,\mathrm{P}^{-1} \qquad (4.9)$$

Substituting (4.9) into the system of equations (4.6) we find $\begin{pmatrix} u_x \\ v_x \end{pmatrix} + P\Lambda P^{-1}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1p_y}{\rho u} - \frac{g}{u} \end{pmatrix}$, that is,

$$P^{-1}\begin{pmatrix} u_x \\ v_x \end{pmatrix} + \Lambda\left(P^{-1}\begin{pmatrix} u_y \\ v_y \end{pmatrix}\right) = P^{-1}\begin{pmatrix} 0 \\ -\frac{1p_y}{\rho u} - \frac{g}{u} \end{pmatrix}$$, that is,

$$\begin{pmatrix} 1 & -\frac{u}{v} \\ 0 & \frac{u}{v} \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{u}{v} \end{pmatrix}\begin{pmatrix} 1 & -\frac{u}{v} \\ 0 & \frac{u}{v} \end{pmatrix}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 1 & -\frac{u}{v} \\ 0 & \frac{u}{v} \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1p_y}{\rho u} - \frac{g}{u} \end{pmatrix}$$, that is

$$\begin{pmatrix} u_x - \frac{uv_x}{v} \\ \frac{uv_x}{v} \end{pmatrix} + \begin{pmatrix} 0 \\ v_y \end{pmatrix} = \begin{pmatrix} \frac{1}{v}\left(\frac{1p_y}{\rho} + g\right) \\ \frac{1}{v}\left(-\frac{1p_y}{\rho} - g\right) \end{pmatrix}$$, so that

so that
$$u_x - \frac{uv_x}{v} = \frac{1}{v}\left(\frac{1}{\rho}p_y + g\right) \qquad (4.10)$$

$$\frac{uv_x}{v} + v_y = \frac{1}{v}\left(-\frac{1}{\rho}p_y - g\right) \qquad (4.11)$$

Equation (4.11) is a linear first order partial differential equation in v and the associated Lagrange's equation is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dv}{-\frac{1}{\rho}P_y - g} \qquad (4.12)$$

From (4.12) we obtain the ordinary differential equation $vdv = \left(-\frac{1}{\rho}p_y - g\right)dy$ which gives on integration

$$\frac{1}{2}v^2 = -\frac{1}{\rho}p_y - gy + c_1 \qquad (4.13)$$

where c_1 is an arbitrary constant . The expression for v is $v = \sqrt{2c_1 - \frac{2p}{\rho} - 2gy}$.

At the free surface, $y = h(x)$, $p = p^*$, where p* is the excess pressure over the static pressure, and v_h is the velocity of the fluid at the free surface. We then have from (4.13)

$$\frac{1}{2}v_h^2(x,h) = -\frac{1}{\rho}P^* - gh + c_1 \quad \text{or} \quad c_1 = \frac{1}{2}v_h^2(x,h) + \frac{1}{\rho}p^* + gh \qquad (4.14)$$

Substituting (4.14) into (4.13) we find $\frac{1}{2}v_2^2 - \frac{1}{2}v_h^2 = \frac{p^*}{\rho} + gh - \frac{p}{\rho} - gy$

The pressure equation is $P = P^* + \rho g(h-y) + \frac{1}{2}\rho v_h^2 = \frac{1}{2}\rho v^2$ (4.15)

Substitution of (4.15) into (4.10) gives

$$vu_x - uv_x = \frac{1}{\rho}\frac{\partial p^*}{\partial y} - \frac{v\partial v}{\partial y}^2 \qquad (4.16)$$

But $vu_x - uv_x = -u^2\frac{\partial}{\partial x}\left(\frac{v}{u}\right)$ so that (4.16) becomes

$$u_2\frac{\partial}{\partial x}\left(\frac{v}{u}\right) = -\frac{\partial}{\partial y}\left(\frac{1}{\rho}p^* - \frac{1}{2}v^2\right) \qquad (4.17)$$

Equation (4.17) is a new form of momentum equation derived using the method of characteristics. Its form differs from that given in [1].

The continuity condition across a stationary jump in the flow is given by

$$[uh] = 0 \qquad (4.18)$$

Thus $uh = u_1h_1 = u_2h_2$ (4.19)

The second equation that emanates from the Lagrange's equation (4.12) is $\frac{dx}{u} = \frac{dy}{v}$

or

$$v(x,y) = u(x,y)\frac{dy}{dx} \qquad (4.20)$$

At the free surface y = h, so that equation (4.20) becomes

$$v(x,h) = u(x,h)\frac{dh}{dx} \qquad (4.21)$$

Equation (4.21) can further be written as $v(x,h) = uh\frac{dh}{dx}\frac{1}{h}$ (4.22)

Application of non linear velocity profile

We consider a nonlinear velocity profile of the form

$$v(x,y) = v(x,h)\left(\frac{2y^2}{h^2} - \frac{y}{h}\right) \qquad (4.23)$$

Substituting (4.21) into (4.23) we find $v(x,y) = uh\frac{dh}{dx}\frac{1}{h}\left(\frac{2y^2}{h^2} - \frac{y}{h}\right)$ (4.24)

Using (4.19) in (4.24) gives $v = u_1h_1\frac{dh}{dx}\frac{1}{h}\left(\frac{2y^2 - yh}{h^2}\right)$ (4.25)

Substituting (4.19) and (4.25) into (4.17) we obtain

$$\frac{u_1^2h_1^2}{h^2}\frac{\partial}{\partial x}\left(\frac{u_1h_1\frac{dh}{dx}\frac{1}{h}\left(\frac{2y^2 - yh}{h^2}\right)}{\frac{u_1h_1}{h}}\right) = -\frac{\partial}{\partial y}\left(\frac{1}{\rho}p^* - \frac{v^2}{2}\right) \qquad (4.26)$$

Simplification of (4.26) yields.

$$-\frac{1}{\rho}\frac{\partial p^*}{\partial y}=\frac{2u_1^2h_1^2y^2}{h^2}\frac{d}{dx}\left(\frac{1dh}{h^2dx}\right)-\frac{u_1^2h_1^2y^2}{h^2}\frac{d}{dx}\left(\frac{1dh}{hdx}\right)-\frac{8u_1^2h_1^2y^3}{h^6}\left(\frac{dh}{dx}\right)^2$$
$$+\frac{6u_1^2h_1^2y^2}{h^5}\left(\frac{dh}{dx}\right)^2-\frac{u_1^2h_1^2y}{h^4}\left(\frac{dh}{dx}\right)^2 \qquad (4.27)$$

Integration of (4.27) with respect to y from y = 0 to h yields the excess bottom pressure p* given by

$$p^*_{\text{bottom}}=-2\rho\frac{u_1^2h_1^2h}{3}\frac{d}{dx}\left(\frac{1dh}{h^2dx}\right)+\frac{1}{2}\rho u_1^2h_1^2\frac{d}{dx}\left(\frac{1dh}{hdx}\right)+\frac{2\rho u_1^2h_1^2}{h^2}\left(\frac{dh}{dx}\right)^2-\frac{2\rho u_1^2h_1^2}{h^2}\left(\frac{dh}{dx}\right)^2+\frac{\rho u_1^2h_1^2}{2h^2}\left(\frac{dh}{dx}\right)^2$$

which on simplification gives

$$p^*_{\text{bottom}}=-2\rho\frac{u_1^2h_1^2h}{3}\frac{d}{dx}\left(\frac{1dh}{h^2dx}\right)+\frac{1}{2}\rho u_1^2h_1^2\frac{d}{dx}\left(\frac{1dh}{hdx}\right)+\frac{\rho u_1^2h_1^2}{2h^2}\left(\frac{dh}{dx}\right)^2 \qquad (4.28)$$

Multiplying (4.28) by x and integrating the result with respects to x in the interval ($-a$, a) gives the torque per unit transverse distance exerted by the excess bottom pressure on the fluid in the clockwise direction. Thus, by (4.1) this torque becomes

$$T=\frac{t^*}{\rho u_1^2h_1^2}=-\frac{2}{3}\int_{-a}^{a}xh\frac{d}{dx}\left(\frac{1dh}{h^2dx}\right)dx+\frac{1}{2}\int_{-a}^{a}x\frac{d}{dx}\left(\frac{1dh}{hdx}\right)dx+\frac{1}{2}\int_{-a}^{a}\frac{x}{h^2}\left(\frac{dh}{dx}\right)^2dx \qquad (4.29)$$

Integrating (4.29) by parts we obtain after simplification

$$T=-\frac{1}{6}\left(\frac{xdh}{hdx}\right)_{-a}^{a}+\frac{1}{6}(\ln h)_{-a}^{a}+\frac{7}{6}\int_{-a}^{a}\frac{x}{h^2}\left(\frac{dh}{dx}\right)^2dx$$

or
$$T=-\frac{1}{6}\left(\frac{xdh}{hdx}-\ln h\right)_{-a}^{a}+\frac{7}{6}\int_{-a}^{a}\frac{x}{h^2}\left(\frac{dh}{dx}\right)^2dx \qquad (4.30)$$

Now, as $x\to+a,\ h\to\text{constant}$ and $\left(\frac{xdh}{hdx}\right)\to 0.$. Similarly, $\left(\frac{xdh}{hdx}\right)\to 0.$ as $x\to-a.$ Thus (4.30) becomes

$$T=-\frac{1}{6}(-\ln h)_{-a}^{a}+\frac{7}{6}\int_{-a}^{a}\frac{x}{h^2}\left(\frac{dh}{dx}\right)^2dx \qquad (4.31)$$

that is,
$$T=\frac{1}{6}\ln\frac{h_2}{h_1}+\frac{7}{6}\int_{-a}^{a}\frac{x}{h^2}\left(\frac{dh}{dx}\right)^2dx \qquad (4.32)$$

or
$$T=\frac{1}{6}\ln H+\frac{7}{6}K \qquad (4.33)$$

where
$$K=\int_{-a}^{a}\frac{x}{h^2}\left(\frac{dh}{dx}\right)^2dx \qquad (4.34)$$

Now, K as defined in (4.34) can only be evaluated when h is known as a function of x

5.0 Determination of the mean vorticity

On using the torque exerted by the bottom pressure distribution, together with the terms arising from horizontal forces and momenta, in an angular momentum balance, we obtain

$$\frac{gh_1^3}{6}+\frac{u_1^2h_1^2}{2}+u_1^2h_1^2T=\frac{gh_2^3}{6}+\frac{u_2^2h_2^2}{2}+\int_0^{h_2}\left(2u_2(u-u_2)+(u-u_2)^2\right)ydy \qquad (5.1)$$

Using (3.1) in (5.1) we find after simplification

$$\frac{gh_1^3}{6}+\frac{u_1^2h_1^2}{2}+u_1^2h_1^2T=\frac{gh_2^3}{6}+\frac{u_2^2h_2^2}{2}+\int_0^{h_2}\left(\omega u_2h_2-2\omega u_2y+\frac{\omega^2h_2^2}{4}+\omega^2y^2-\omega^2h_2y\right)ydy \qquad (5.2)$$

Dividing (5.2) by h_1^3 and evaluating the integral we obtain after simplification

$$\frac{g}{6}+\frac{u_1^2h_1^2}{2h_1^3}+\frac{u_1^2h_1^2T}{h_1^3}=\frac{gh_2^3}{6h_1^3}+\frac{u_2^2h_2^2}{2h_1^3}+\frac{\omega^2h_2^4}{24h_1^3}-\frac{\omega u_2h_2^2}{6h_1^3} \qquad (5.3)$$

that is, $$\frac{g}{6}+\frac{u_1^2}{2h_1}+\frac{u_1^2T}{h_1}=\frac{gH^3}{6}+\frac{u_2^2h_2^2}{2h_1^3}+\frac{\omega^2h_2^4}{24h_1^3}-\frac{\omega u_2h_2^3}{6} \qquad (5.4)$$

Multiplying (5.4) by $\frac{6}{g}$ and using (2.5) we find

$$1+3F+6FT=H^3-\frac{\omega u_2H^3}{g}+\frac{3u_2^2h_2^2}{gh_1^3}+\frac{\omega^2h_2^4}{4gh_1^3} \qquad (5.5)$$

which simplifies to $$\frac{1}{F}+3+6T=\frac{H^3}{F}-\frac{\omega u_2h_1H^3}{u_1^2}+\frac{3u_2^2H^2}{u_1^2}+\frac{\omega^2h_2^4}{4h_1^3u_1^2} \qquad (5.6)$$

Substituting the continuity equation (2.1) into (5.6) and using (2.7) we obtain

$$\frac{1-H^3}{F}+6T=-\frac{\omega h_2}{u_2}+\frac{\omega^2h_2^2}{4u_2^2} \qquad (5.7)$$

or $$\frac{1-H^3}{F}+6T=-\Omega+\frac{\Omega^2}{4} \qquad (5.8)$$

where $$\Omega=\frac{\omega h_2}{u_2} \qquad (5.9)$$

Substituting for T in (5.8) using (4.33) and simplifying we have

$$\Omega^2-4\Omega+4\left(\frac{H^3-1}{F}\right)-4\ln H-28K=0 \qquad (5.10)$$

This is a quadratic equation in Ω which can be solved to give

$$\Omega=2\left(1-\left(1-\frac{H^3-1}{F}+\ln H+7K\right)^{\frac{1}{2}}\right) \qquad (5.11)$$

Solving (3.12) and (5.11) simultaneously, Ω (*F*) and *H* (*F*) can be obtained and their graphs are sketched in Figure 3. As *F* - 1→0, series solutions for Ω and *F* - 1 in increasing powers of *H* - 1 can be obtained. The result is

$$\Omega=\frac{49}{4}K^2+(2-7K)(H-1)+\frac{(18-49K)}{4}(H-1)^2+\frac{(25-14K)}{6}(H-1)^3$$
$$+\frac{(191-42K)}{48}(H-1)^4+\frac{(73-42K)}{60}(H-1)^5+0[(H-1)]^6 \qquad (5.12)$$

and $$F-1=\frac{3}{2}(H-1)+\frac{1}{2}(H-1)^2+0[(H-1)]^3 \qquad (5.13)$$

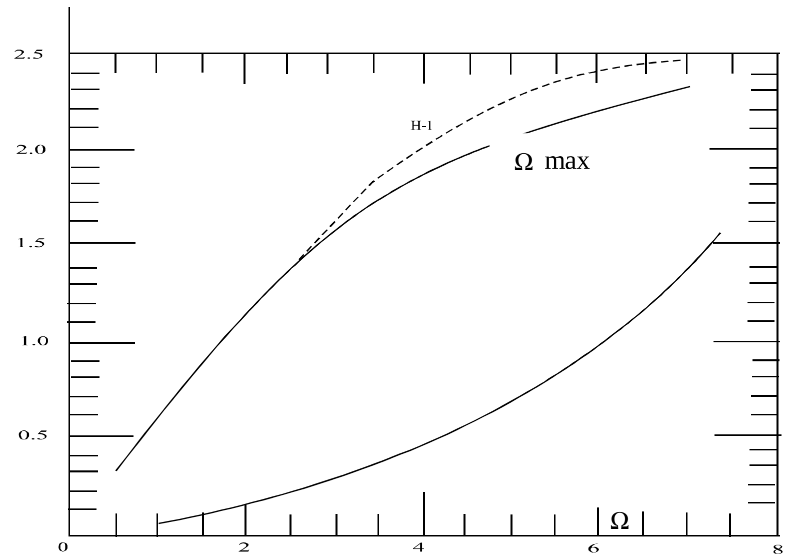

Figure 3: The wave height in the form H-1 and the downstream vorticity Ω as functions of the Froude number (F-1)

In terms of F – 1, (5.12) becomes

$$\Omega = \frac{49}{4}K^2 + \frac{2}{3}(2-7K)(F-1) + \frac{1}{9}(18-49K)(F-1)^2 + \frac{4}{81}(25-14K)(H-1)^3 + 0(F-1)^{46} \qquad (5.14)$$

Thus the vorticity does not increase from zero as a function of the Froude number minus one because of the presence of the first term in (5.14). Hence our result (5.14) generalizes [1]

6.0 Discussion and conclusion

Figure 1 shows a steady-flow hydraulic jump with a control volume surrounding the jump, and upstream and downstream depths together with uniform velocity profiles upstream and downstream of the jump. Using this figure **(see [1])** the classical jump conditions (2.1) and (2.2) were determined. Figure 2 depicts hydraulic jump with control volume for the case with finite mean vorticity downstream of the jump, since the flow downstream of the jump will be obviously rotational at a sufficiently high Froude number. The figure also shows the velocity distribution on the downstream side of the jump given by equation (3.1) with u_2 as the mean velocity. From the figure, a new jump condition, equation (3.2), with downstream vorticity was determined. Figure 3 shows the wave height in the form H-1 and the downstream vorticity Ω as functions of the Froude number minus one (F -1). As may be seen in the figure, H-1 increases qradratically with F -1 near F=1 in the same way as without downstream vorticity. The method of characteristics applied to hydraulic jump was used to determine the mean vorticity downstream of a hydraulic jump. We observe from equation (5.14) that the vorticity does not increase from zero as a function of Froude number minus one because of the presence of the first term in (5.14). Hence our result differs from [1] in that it generalizes [1]. If K = 0 in (5.14), Ω increases from zero in increasing power of H - 1. This is the particular case considered in [1]. In the present solution, it is expected that the physical case corresponds to the term under the square root of equation (5.11) approaching zero smoothly as $F \rightarrow \infty$. In this case, Ω in equation (5.11) will approach the value 2 asymptotically. This is also the maximum value that can be attained by Ω because it corresponds to $u(h_2) = 0$. Larger values of Ω imply negative $u(h_2)$, which does not make sense, because it corresponds to the downstream fluid overtaking the wave. The dashed curve in the figure gives the classical result (the result without downstream vorticity) represented by equation (2.6). The difference between the dashed and full lines for H - 1 remains small as F increases since Ω is limited to 2.

In the case of the non linear velocity profile, we notice that the mean vorticity increases in the increasing power of F-1 (see equation (5.14)), which is different from the prediction of the linear model. We also notice that the torque determined in (4.33) using the non linear profile has a finite value just like the one from the linear model. The difference between the two lies in the fact the torque determined using the non linear profile increases (or decreases) according as the parameter K in (4.33) increases (or decreases).

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