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Self-similar solution for coupled thermal electromagnetic model during microwave heating of biological tissues.

***E. A. Adebile, B. N. Akintewe, O. K. Olaleye,. and V. Idoko**
Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria.
***e-mail: eaadebiletri@yahoo.com.**

Abstract

An investigation into the existence and uniqueness solution of self-similar solution for the coupled Maxwell and Pennes Bio-heat equations have been done. Criteria for existence and uniqueness of self-similar solution are revealed in the consequent theorems

***Keywords*:** Self-similar variable, Maxwell equation, Pennes Bio-heat equation, Microwave heating, biological tissues.

1.0 Introduction

The use of microwave radiation has received great boost in the recent years for industrial applications. It's been used for solutions such as smelting, sinistering and drying. It has been found that it has many applications in joining mathematical and medical fields or especially in clinical cancer therapy hyperthermia (Hill and Pincombe, 1992; Deuflhand and Seebass, 1998). Despite the usefulness of these devices there has been an increase in public concern about the possible health risk from electromagnetic energy emitted from various sources. Because the increasing use of radio frequency (RF) energy, it is becoming even more important to identify the limit of safe exposure with respect to thermal hazards.

Many researchers here studied and reported in the literature the analysis of thermal effects, which have been done with model studies. A number of the work done took physically reasonable assumptions to provide answers to the public concern on health risks. Very few solved their model analytically while a great number used numerical method such as finite difference method, finite element method and so on. Some of the works done are: Jiang et al (2002) discussed the effects of thermal properties and geometrical dimensions on the skin burn injuries.

Merchant and Liu (2001) investigated the steady state microwave heating of finite one-dimensional slab. They took the electrical conductivity and thermal absorptivity to be dependent on temperature and they are considered to be governed by the Arrhenius law, while the electrical permittivity and permeability were assumed constant. The simulation of the thermal wave propagation in biological tissues by dual reciprocity boundary element method has been considered by Liu et al (1998). Adebile (1997) and Adebile et al (2004) and Adebile and Ogunmoyela here studied the preferential heating of cancerous biological tissue and sparing of normal tissues for different assumptions taken in respect of the tissue parameters and the heating device. Asymptotic method of analysis and the use of series method have been the procedure used to seek for solution for the governing equations obtained.

Hill and Pincombe (1992) in their model to involve the Maxwell equation coupled with the heat equation. They introduced various similarity temperature profiles, for different assumptions in the model parameters. They obtained some closed analytical expressions but in general the resulting ordinary differential equation were solved numerically. Although their work does not put into account the microcirculatory effect of blood because of their problem. Alao (2002) in his work on viscous reacting flow investigated self-similar solution for their mathematical model.

Very recently El-dabe et al (2003) using a finite difference technique solved the coupled Maxwell's Penne's bio-heat equations for a special case when $m=1$. The purpose of this paper is to investigate the self-similar solution of the coupled Maxwell's and penne bio-heat equation for a generalized body-heating coefficient of the form $Q(T)=T^m$ El-dabe et al proffered a numerical solution for the case $m=1$.

2.0 Mathematical formulation

(Hill and Pincombe, 1992) reported in their paper that the heat source arising from microwave irradiation in proportional to the square of the modules of the electric field intensity. We thus consider for our governing equations the following as in El-dabe et al (2003).

$$\frac{\partial \mathrm{E}}{\partial x}+\in\frac{\partial \mathrm{E}}{\partial t}+\partial\,\mathrm{E}=0 \qquad (2.1)$$

$$\frac{\partial \mathrm{E}}{\partial x}+\mu\frac{\partial \mathrm{H}}{\partial t}=0 \qquad (2.2)$$

$$\sigma c_p\frac{\partial \mathrm{T}}{\partial t}=\frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{T}}{\partial x}\right)-\omega_b\alpha_b c_b\left(\mathrm{T}-\mathrm{T}_b\right)+Q\left(\mathrm{T}\right)\left|\mathrm{E}\right|^2 \qquad (2.3)$$

With the initial and boundary conditions:

$$\mathrm{T}(x,0)=\frac{\mathrm{T}c}{L}x;\mathrm{T}(0,t)=0;\mathrm{T}(L,t)=\mathrm{T}c,\quad E(x,0)=\frac{E_0}{L}x;E(0,t)=0;E(L,t)=E_0$$

$$\mathrm{H}(x,0)=\frac{\mathrm{H}_0}{L}x;\mathrm{H}(0,t)=0;\mathrm{H}(I,t)=\mathrm{H}_0 \qquad (2.4)$$

El-dabe et al (2003) solved the above system of equation numerically for m = 1 assuming that the body heating coefficient has the form $Q(\mathrm{T})=\mathrm{T}^m, m\geq 1$ (2.5)

This assumption is in line with the research literature as in merchant and Liu (2001), that the physical properties of material have power law dependence on temperature. Define the following dimensionless variables as in El-dabe et al (2003)

$$\mathrm{T}=\frac{\tau v}{L^2};\eta=\frac{x}{L},\theta=\frac{\mathrm{T}}{\mathrm{T}_b};c_1=\frac{c_b}{c_p},\ \overline{\mathrm{E}}=\frac{\mathrm{E}}{\mathrm{E}_0};\overline{\mathrm{H}}=\frac{\mathrm{H}}{\mathrm{H}_0},\mathrm{P}_1=\frac{\mathrm{P}_b}{\mathrm{P}}'\lambda_1=\frac{v\varepsilon\mathrm{E}_0}{L\mathrm{H}_0}$$

$$\lambda_2=\frac{L\sigma\mathrm{E}_0}{\mathrm{H}_0};\lambda_3=\frac{\mu\mathrm{H}_0 v}{LE_0},\lambda\frac{L^2\mathrm{T}_b^{m-1}\left|\mathrm{E}_0\right|^2}{vpc_p} \qquad (2.6)$$

Using these dimensionless variables in equation (2.1) – (2.4) are:

$$\frac{\partial \mathrm{H}}{\partial\eta}+\lambda_1\frac{\partial \mathrm{E}}{\partial\tau}+\lambda_2\,\mathrm{E}=0 \qquad (2.7)$$

$$\frac{\partial \mathrm{E}}{\partial\eta}+\lambda_3\frac{\partial \mathrm{H}}{\partial\tau}=0 \qquad (2.8)$$

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{\mathrm{P}_r}\frac{\partial^2\theta}{\partial\eta^2}-\omega_1 p_1 c_1\left(\theta-1\right)+\lambda Q\left|\mathrm{E}\right|^2 \qquad (2.9)$$

Subject to the following dimensionless initial and boundary conditions,

$$\theta(\eta,0)=\frac{\mathrm{T}_c}{\mathrm{T}_b}\eta;\ \theta(0,\tau)=0;\quad \theta(1,\tau)=\frac{\mathrm{T}_c}{\mathrm{T}_b},\ \mathrm{E}(\eta,0)=\eta;\quad \mathrm{E}(0,\tau)=0;\quad \mathrm{E}(1,\tau)=1$$

$$\mathrm{H}(\eta,0)=\eta;\quad \mathrm{H}(0,\tau)=0;\quad \mathrm{H}(1,\tau)=1 \qquad (2.10)$$

3.0 Method of solution

We seek a self-similar solution of the form

$$\xi=x^p\,t^q \qquad (3.1)$$

$$\mathrm{T}(\eta,\tau)=f(\xi);\mathrm{E}(\eta,\tau)=h(\xi);\mathrm{H}(\eta,\tau)=q(\xi) \qquad (3.2)$$

Using the relations in (3.1) and (3.2) in equations (2.7) – (2.10) we have

$$\frac{\partial \mathrm{T}}{\partial\tau}=q\eta^p\tau^{p-1}\frac{df}{d\xi}$$

$$\frac{\partial \mathrm{T}}{\partial\eta}=p\eta^{p-1}\frac{df}{d\xi}$$

$$\frac{\partial^2 \mathrm{T}}{\partial \eta^2} = \mathrm{P}^2 \eta^{2(p-1)} \tau^{2q} \frac{d^2 f}{d\xi^2} \tag{3.3}$$

$$\mathrm{P}\eta^{\mathrm{P}-1}\tau^q \frac{dg}{d\eta} + \lambda_1 q\eta^p \tau^{q-1} \frac{dh}{d\eta} + \lambda^2 h = 0 \tag{3.4}$$

$$q\eta^p \tau^{q-1} \frac{dg}{d\eta} + \lambda_3 \mathrm{P}\eta^{p-1}\tau^q \frac{dh}{d\eta} = 0 \tag{3.5}$$

$$q\eta^p \tau^{q-1} \frac{df}{d\eta} = \Omega p^2 \eta^{2(p-1)} \tau^{2q} \frac{d^2 f}{d\eta^2} - \alpha(f-1) + \lambda h^2 f^m \tag{3.6}$$

Using the $\eta^p = \xi \tau^{-q}$ (3.7)

In equations (3.4) – (3.6), we have respectively the equations:

$$\mathrm{P}\eta^{2(p-1)} \tau^{-2pq+3q} \frac{dg}{d\eta} + \lambda_1 q\eta^p \tau^{-1} \frac{dh}{d\eta} + \lambda_2 h = 0 \tag{3.8}$$

$$q\eta \tau^{-1} \frac{dg}{d\eta} + \lambda_3 \eta^{p-1} \tau^{pq} \frac{dh}{d\eta} = 0 \tag{3.9}$$

$$q\eta \tau^{-1} \frac{df}{d\eta} = \Omega \mathrm{P}^2 \eta^{2(p-1)} \tau^{-2pq+4q} \frac{d^2 f}{d\eta^2} - \alpha(f-1) + \lambda h^2 f^m \tag{3.10}$$

For self-similarity we set $q = 0; -2pq + 3q = 0; -2pq + 4q = 0$

Then $2pq = 0$ (3.11)

Using the conditions (3.11) in (3.8) – (3.10) becomes:

$$\mathrm{P}\eta^{2(p-1)} \frac{dg}{d\eta} + \lambda_2 h = 0 \tag{3.12}$$

$$\lambda_3 \eta^{p-1} \frac{dh}{d\eta} = 0 \tag{3.13}$$

$$\frac{d^2 f}{d\eta^2} = \frac{\alpha(f-1) - \lambda h^2 f^m}{\Omega P \eta^{2(P-1)}} \tag{3.14}$$

With the result and boundary conditions , $f(0)=0;\ f(1)=\frac{\mathrm{T}_c}{\mathrm{T}_b}$

$$g(0)=0;\ g(1)=1 \tag{3.15}$$

$h(0)=0;\ h(1)=1$

Let

$$x_1 = \eta, x_2 = f,\ x_3 = f^1, x_4 = h, x_5 = g \tag{3.16}$$

Therefore

$x_1^1 = 1$

$x_2^1 = f^1$

$x_3^1 = f^{11} = \frac{\alpha(f-1) - \lambda h^2 f^m}{\Omega \mathrm{P}\eta^{2(p-1)}}$

$x_4^1 = o$

$$x_5^1 = \frac{-\lambda_2 h}{\mathrm{P}\eta^{2(p-1)}} \tag{3.17}$$

The relation in (3.17) can be represented thus:

$$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \\ x_5^1 \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ \dfrac{\alpha(x_2-1)+\lambda x_4^2 x_2^m}{\Omega \mathrm{P}^2 x_1^{2(p-1)}} \\ 0 \\ \dfrac{-\lambda_2 x_4}{\mathrm{P} x_1^{2(p-1)}} \end{pmatrix} \qquad (3.18)$$

Satisfying the initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\alpha \\ 0 \\ 0 \end{pmatrix} \qquad (3.19)$$

Equation (3.18) and (3.19) is equivalently written as

$$x^1 = f(x_1, x_2, x_3, x_4, x_5), \quad x(0) = x_0 \qquad (3.20)$$

That is

$$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \\ x_5^1 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, x_3, x_4, x_5) \\ f_2(x_1, x_2, x_3, x_4, x_5) \\ . \\ . \\ f_5(x_1, x_2, x_3 \ldots\ldots x_5) \end{pmatrix} \qquad (3.21)$$

4.0 Existence and uniqueness of solution;

Let D: $0 \le x, < L$; $0 \le x_2 < N_1$;$-N_2 \le x_3 \le 0$; $0 \le x_4 < N_3$ $0 \le x_4$ N_5

***Theorem* 3.1**

Let S hold, then for $0 \le \lambda_2 < \infty$, $0 \le \lambda < \infty$, $0 \le \Omega < \infty$, $0 \le \alpha < \infty$, $1 \le p < \infty$, $m \ge 0$, there exist a unique solution for the equation (3.18) satisfying (3.19)

Proof

The equation (3.19) can be represented s

$$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \\ x_5^1 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, x_3, x_4, x_5) \\ f_2(x_1, x_2, x_3, x_4, x_5) \\ . \\ . \\ f_5(x_1, x_2, x_3 \ldots\ldots x_5) \end{pmatrix} \qquad (4.1)$$

From this above relation we have

$$\left|\frac{\partial f_1}{\partial xj}\right| \le 0, \; j = 1\,(\;)\,5.$$

$$\left|\frac{\partial f_2}{\partial xj}\right| \le 0,\ j=1,3,4,5;\quad \left|\frac{\partial f_2}{\partial x_2}\right| \le 1$$

$$\left|\frac{\partial f_3}{\partial x_1}\right| = \left|\frac{2(p-1)x_1^{2(1-p)-1}\{\alpha(x_2-1)+\lambda x_4^2 x_2^m\}}{\Omega p^2}\right| \le \frac{2(p-1)(\alpha+\delta_1\lambda\omega)}{\Omega p^2}$$

Where $\delta=(mex\ x_4),\ \delta_1=\ square(0\ mex\ x_4)$

$$\left|\frac{\partial f_3}{\partial x_2}\right| = \left|\frac{x_1^{2(1-p)}(\alpha+m\lambda_4^2 x_2^{m-1}}{\Omega p^2}\right| \le \frac{\alpha+\delta_1 m\lambda\omega}{\Omega p^2}$$

where $\omega=(mexx_2)^m$

$$\left|\frac{\partial f_3}{\partial x_3}\right| \le 0,\quad \left|\frac{\partial f_3}{\partial x_5}\right| \le 0$$

$$\left|\frac{\partial f_3}{\partial x_4}\right| = \left|\frac{2x_1^{2(1-p)}\lambda x_4 x_2^m}{\Omega p^2}\right| \le \frac{2\lambda\delta}{\Omega p^2} \text{ provided } \left(2(1-p)\le 0,\ x_4 \le 1m \le 0\right)$$

$$\left|\frac{\partial f_4}{\partial x_1}\right| \le 0$$

$$\left|\frac{\partial f_5}{\partial x_1}\right| = \left|\frac{2p\lambda x_4(p-1)x_1^{2(p-1)-1}}{p^2 x_1^{4(p-1)}}\right| = \left|\frac{2(p-1)\lambda_2 x_4\ x_1^{2(1-p)-1}}{p}\right| \le \frac{2(p-1)\lambda_2\sigma}{p} \text{ provided } \left(2(1-p)-1\le 0\ \&\ x_4\le 1\right)$$

$$\left|\frac{\partial f_5}{\partial x_2}\right| \le 0,\ \left|\frac{\partial f_5}{\partial x_3}\right| \le 0\ ,,\left|\frac{\partial f_5}{\partial x_4}\right| = \left|\frac{-p\,x_1^{2(p-1)}\lambda_2}{p^2 x^{4(p-1)}}\right| = \left|\frac{-x_1^{2(1-p)}\lambda_2}{p^2}\right| \le \frac{\lambda_2}{p^2} \text{ provided } 2(1-p)\le 0$$

$$\left|\frac{\partial f_5}{\partial x_5}\right| \le 0,\quad \left|\frac{\partial f_5}{\partial x_4}\right| = \left|\frac{-px_1^{2(p-1)}\lambda_2}{p^2 x^{4(p-1)}}\right| = \left|\frac{-x_1^{2(1-p)}\lambda_2}{p^2}\right| \le \frac{\lambda_2}{p^2} \text{ provided } 2(1-p) \le 0)$$

$\left|\frac{\partial f_5}{\partial x_5}\right| \le 0$. Therefore $\frac{\partial fi}{\partial xj}, i, j$ is bounded, and there exist K $\left(where\,\mathrm{K}=\max\ (k_{ij})\ \right)$ such that

$\left|\frac{\partial fi}{\partial xj}\right| \le \mathrm{K}\ , where\ o < K < \infty$. Therefore $f_i\left(x_1, x_2, x_3, ...x_5\right),\ \ i=1.....5$ are Lipchitz continuous for different conditions on m, p $\lambda_2, \lambda_3\ \lambda, \alpha, and\ \Omega$. Hence there exist a unique solution of the system of equations (2.33) and (2.34).

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