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The period of relaxation oscillations of a nonlinear system using singular perturbation methods

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Abstract

We determine the period of relaxation oscillations of a physical system governed by the nonlinear Liénard equation $\varepsilon x'' + (ax^2 - b)x' + x + cx^3 = 0$ where $a, b, c > 0$, $0 < \varepsilon << 1$, using singular perturbation methods. These methods which involve considering matched asymptotic expressions of different layers yield a uniformly valid expansion for the above equation and hence the relaxation oscillations. The van der Pol equation is a special case of the above equation.

1.0 Introduction

A number of interesting systems of mathematical physics occur which require asymptotic analysis of their governing equations. Notable among these are systems which are governed by the non-linear differential equation

$$\varepsilon x' = f(x', x) \qquad (1.1)$$

The question is what happens with the periodic solutions of (1.1) as $\varepsilon \to 0$, in particular if the limit equation

$$f(x', x) = 0 \qquad (1.2)$$

has no periodic solution. Of course, there could be no boundary layer effect in the strict sense since there is no boundary.

A problem of this type was first treated by van der Pol in 1927 who explained using phase plane analysis the occurrence of certain "*jerky* (or *relaxation*) *oscillations*" in electric networks. Subsequently much work was done on electric and mechanical oscillations of this kind.

Results on asymptotic periodic solutions have been obtained by Levinson in 1942. Other examples of this kind of systems are in the gas. In this paper, we use singular perturbation methods to determine the period of relaxation oscillations of a non-linear system prescribed by a form of the Liénard equation

$$\varepsilon x'' + (ax^2 - b)x' + x + cx^3 = 0, \quad 0 < \varepsilon << 1, \ a, b, c, > 0. \qquad (1.3)$$

Singular perturbation problems have been discussed notably by J. D. Cole (1968), M. Van Dyke (1964) and H. Grabmuller (1978) among others. In fact, Cole (1968) applies the method in solving the problem of the van der Pol oscillator.

A singular perturbation problem could be defined as one in which no single asymptotic expansion is uniformly valid throughout the field of interest. The process therefore involves getting the asymptotic expansion of the three layers involved at a discontinuity. We generalize the procedure of Cole (1968) to obtain the period of oscillations by matching as proposed by Prandtl and successfully applied by van Dyke (1964) and a host of others. Duccio Papini (1999) discusses the existence of periodic solutions of a different class of Liénard equations which also covers our class.

The resulting asymptotic expansions are matched to obtain a uniformly valid expansion for the given non-linear differential equation and hence obtain the period of oscillations of the system comparing same with results obtained by phase plane methods as described by Bogoliubov and Mitropolsky (1961), Stoker (1950) and La Salle (1949).

We have not bothered to prove the existence and uniqueness of a limit cycle for the given non-linear ordinary differential equation (which is fundamental to the occurrence of relaxation oscillations) since this abounds in the literature.

2.0 Asymptotic xxpansion of the outer and inner layers

Relaxation oscillations are periodic motions with nearly discontinuous systems. There are two phases of the physical system, namely, a fast phase and a slow phase. And in each phase different physical phenomena dominate. In this section we shall use the method of matched asymptotic expansions similar to the procedure of Cole (1968) for the Van der Pol oscillator. Consider the ordinary differential equation

$$\varepsilon x'' + \left(ax^2 - b\right)x' + x + cx^3 = 0 \tag{2.1}$$

where $a, bc > 0$, $0 < \varepsilon << 1$, and $(')$ means $\frac{d}{dt}$, etc.

The *outer* expansion is associated with the limit process $\in \to$ o, t fixed and has the form

$$x(t;\varepsilon) = x_\circ(t) + \varepsilon x_1(t) + \ldots \tag{2.2}$$

and substituting this into (2.1) and equating coefficients of like power of ε, we have

$$\left(ax_\circ^2 - b\right)x_\circ' + x_\circ^3 + = 0 \tag{2.3}$$

$$2ax_\circ x_\circ' x_1 + \left(ax_\circ^2 - b\right)x_1' + x_1 + 3cx_\circ^2 x = -x_\circ'' \tag{2.4}$$

From (2.3) $$\frac{dx_\circ}{dt} = \frac{x_\circ + cz_\circ^3}{b - ax_\circ} \tag{2.5}$$

This is a separable ordinary differential equation and is readily solved to get

$$b\log x_\circ - \frac{a+bc}{2c}\log\left(1 + cx_\circ^2\right) = t + k_1 \tag{2.6}$$

(k_1 is a constant of integration).

From (2.5), we find that when $dx_0/dt \to \infty$, $x_0 \to \pm\sqrt{\frac{b}{a}}$ and since the choice of the time origin is arbitrary, we shift the curves so that the slope, dx_0/dt is ∞ at $t = 0$, that is, $x_0 = -\sqrt{\frac{b}{a}}$ for $t = 0$ [see Figure 1.1]

Thus, (2.6) yields $k_1 = b\log\sqrt{b/a} - \frac{a+bc}{2c}\log\left(a + \frac{bc}{a}\right)$, and (2.6) becomes

$$b\log\left(x_{\circ/}\sqrt{b/a}\right) - \frac{a+bc}{2c}\log\left(\frac{1+cx_\circ^2}{1+bc/a}\right) = t \tag{2.7}$$

We now need the behaviour of x_0 near $t = 0$ and for simplicity we let $y_\circ = \frac{x_\circ}{\sqrt{b/c}}$ (scaling) (2.7) therefore becomes

$$b\log y_\circ - \frac{a+b}{2c}\log\left(\frac{1+cy_\circ^2}{1+bc/aa}\right) = t \tag{2.8}$$

As $t \to 0^-, y_\circ \to 1$ we write $t = \alpha_1(y_\circ - 1) + \alpha_2(y_\circ - 1)^2 + \cdots$ and let $y_\circ - 1 = \mu$ so that

$$t = \alpha_1\mu + \alpha_1\mu + \alpha_2\mu^2 + \cdots$$

We now seek to write (2.8) in terms of μ and thence obtain μ in terms of t by inverting the series. Now in (2.8)

$$\log y_\circ = \log(1+\mu) = \mu - \tfrac{1}{2}\mu^2 + \tfrac{1}{3}\mu^3 + O\left(\mu^4\right)$$

and $$\log\left[\frac{1+bcy_\circ^2/a}{1+bc/a}\right] = \log\left(\frac{1+bc\left(1+2\mu+\mu^2\right)}{1+bc/a}a\right) = t$$

$$= \log\frac{2bc}{a+bc}\mu + \left(\frac{bc}{a+bc} - \frac{2(bc)^2}{(a+bc)^2}\right)\mu^2 + \left[\frac{8}{3}\left(\frac{bc}{a+c}\right)^3 - 2\left(\frac{bc}{a+bc}\right)^2\right]\mu^3 + \ldots$$

On simplification (2.8) then becomes

$$-\frac{ab}{a+bc}\mu^2+\frac{ab(a+5bc)}{3(a+bc)^2}\mu^3+0(\mu^4)=t \qquad (2.9)$$

Noting that since $t=0(\mu^2)$ as $\mu\to 0$, then $\mu=(\sqrt{-t})$ as $t\to 0$. That is, we can reverse the series to get

$$\mu=\mu_1\sqrt{-1}+\mu_2(-t)+\mu_3(-t)^{3/2}+\ldots \qquad (2.10)$$

which together yields $\mu_1=\sqrt{\frac{a+bc}{ab}}$

$$-\frac{2ab}{a+bc}\mu_1\mu_2+\frac{ab(a+5bc_2)}{3(a+bc)}\mu_1^3=0$$

$$-\frac{ab}{a+bc}\left[\mu_2^2+2\mu_1\mu_3\right]+\frac{ab(a+5bc)}{3(a+bc)^2}3\mu_1^2\mu_2=0$$

and solving the last two equations yields $u^2=\frac{a+5bc}{6ab},\ \mu^3=\frac{5}{15}\left(\frac{a+5bc}{ab}\right)^2\sqrt{\frac{ab}{a+bc}}$

Using these results in (1.10) and noting that $x_0=\sqrt{\left(\frac{b}{a}\right)}1(1+\mu)$ we obtain

$$x_\circ=\sqrt{\frac{b}{a}}\left\{1+\sqrt{\frac{a+bc}{ab}}\sqrt{-t}+\frac{a+5bc}{6ab}(-1)+\frac{5}{72}\left(\frac{a+5bc}{ab}\right)^2\sqrt{\frac{ab}{a+bc}}(-t)^{3/2}\right\}+0\left[(-t)^{5/2}\right] \qquad (2.11)$$

To find x_1 we consider equation (2.4), which can be put in the form

$$\frac{d}{dt}\left[(b-ax_\circ^2)\right]-x_1\left[1+3cx_\circ^2\right]=\frac{d^2x_\circ}{dt} \qquad (2.12)$$

It is convenient to consider x_1 as a function of x_0 and rewrite (2.12), after some simplification, as

$$\frac{d(ax_\circ^2-b)}{dx_\circ x_\circ+cx_\circ}x_1=-\frac{2ax_\circ}{(ax_\circ^2-b)^2}-\frac{1}{bx_\circ}+\frac{a(a+3bc)x_\circ}{b(a+bc)(ax_\circ^2-b)}-\frac{2c^2x_\circ}{(a+bc)(1+cx_\circ^2)} \qquad (2.13)$$

Integrating (2.13), with respect to x_0, gives

$$x_1=\frac{x_\circ(1+cx_\circ{}^2)}{ax_\circ^2-b}\left\{\frac{1}{ax_\circ^2-b}\frac{\log x_\circ}{b}+\frac{a+3bc}{2b(a+bc)}\log(ax_\circ^2-b)+k_2-\frac{c}{a+bc}\log(1+cx_\circ^2)\right\} \qquad (2.14)$$

where k_2 is a constant of integration. For convenience, let $\alpha_\circ=\sqrt{b/a},\alpha_1=(1/a)\sqrt{a+bc}$

$$\alpha_2=\frac{a+5bc}{6ab}\sqrt{\frac{b}{a}},\alpha_3=\frac{5}{72}\left(\frac{a+5bc}{ab}\right)^2\sqrt{\frac{b}{a}}$$

Then (1.11) can be written in the form

$$x_\circ=\alpha_\circ+\alpha_1\sqrt{-t}+\alpha_2(-t)+\alpha_3(-t)^{3/2}+0\left((-t)^{5/2}\right) \qquad (2.15)$$

so that

$$\log(ax_\circ^2-b)=\log 2a\alpha_\circ\alpha_1+\log(-t)^{1/2}+\frac{2\alpha_0\alpha_2+\alpha_1^2}{2\alpha_0\alpha_1}(-t)^{1/2}-\left[\tfrac{1}{2}\left(\frac{2\alpha_\circ\alpha_2+\alpha_1^2}{2\alpha_\circ\alpha_1}\right)^2-\frac{\alpha_\circ\alpha_3+\alpha_1\alpha_2}{2\alpha_\circ\alpha_1}\right](-t)+\ldots$$

$$\frac{1}{ax_\circ^2-b}=\frac{1}{2a\alpha_\circ\alpha_1(-t)^{1/2}}\left\{1-\frac{2\alpha_\circ\alpha_2+\alpha_1^2(-t)^{1/2}}{2\alpha_0\alpha_1}+\left[\left(\frac{2\alpha_\circ\alpha_2+\alpha_1^2}{2\alpha_0\alpha_1}\right)^2-\frac{\alpha_\circ\alpha_3+\alpha_1\alpha_2}{\alpha_\circ\alpha_1}\right](-t)+\ldots\right\}$$

Therefore (2.14) becomes

$$x_1=\frac{1+c\alpha_\circ^2}{4a^2\alpha_\circ\alpha(-t)}+\frac{(1+c\alpha_\circ^2)(a+3bc)}{4ab\alpha_1(a+bc)}\frac{\log(-t)^{1/2}}{(-t)^{1/2}}+\frac{1}{4a\alpha_\circ\alpha_1}\left\{\frac{A}{2a\alpha_\circ\alpha_1}+\alpha_\circ(1+c\alpha_\circ^2)\left[\frac{(a+3bc)}{2b(a+bc)}\log 2c\alpha_\circ\alpha_1\right.\right.$$

$$\left.\left.-\frac{2\alpha_\circ\alpha_2+\alpha_1^2}{4a\alpha_\circ^2\alpha_1^2}-\frac{1}{b}\log\alpha_\circ+\frac{k_2C}{a+bc}\log(1+c\alpha_\circ^2)\right]\right\}\frac{1}{(-t)^{1/2}}+\ldots$$

where $$A=\left(1+c\alpha_\circ^2\right)\alpha_1+2\alpha_\circ^2\alpha_1 c-\frac{1}{2\alpha_1}\left(1+c\alpha_\circ^2\right)\left(2\alpha_\circ\alpha_2+\alpha_1^2\right)\ldots \tag{2.16}$$

In terms of the original parameters, *a, b, c*, we have

$$x_1=\frac{1}{4\sqrt{ab}(-t)}+\frac{a+3bc}{4ab\sqrt{a+bc}}\frac{\log(-t)^{1/2}}{(-t)^{1/2}}+\frac{1}{4ab\sqrt{a+bc}}(-t)^{1/2}\left\{\frac{bc-a}{3}+\right.$$
$$\left.+2k_2 b(a+bc)+(a+3bc)\log 2\sqrt{\frac{b(a+bc)}{a}}+2(a+bc)\log\sqrt{b/a}\ -2bc\log\frac{a+bc}{a}\right\}+\ldots \tag{2.17}$$

Equations (2.11) and (2.17) together give the outer expansion of (1.1). We now consider joining the two branches AB and CD Figure (1.1) with a boundary layer whose thickness is 0(ε). Since the time origin is not fixed for this expansion, we consider the limit process.

$$\left(\varepsilon\to 0, t*=\frac{t-\delta(\varepsilon)}{\varepsilon}\ \text{Fixed}\right) \tag{2.18}$$

where δ(ε) is to be determined. For matching to x_0 as t →0, the first term of (2.1) is 0(1). We therefore assume the inner expansion in terms of t* in the form

$$x(t;\varepsilon)=g_0(t*)+\beta_1(\varepsilon)g_1(t*)+\beta_2(t*)+\ldots \tag{2.19}$$

where $\{\beta_n(\varepsilon)\}$, *n* = 1, 2, … is an asymptotic sequence for $\varepsilon\to 0$. Writing (1.1) in terms of t*, we have

$$\frac{d^2x}{dt^{*2}}+\left(ax^2-b\right)\frac{dx}{dt*}+\varepsilon\left(x+cx^3\right)=0;\varepsilon\to 0 \tag{2.20}$$

Substituting the expansion (2.19) into (2.20) and equating terms of order 1 and $\beta_1(\varepsilon)$ respectively, we obtain

$$\left.\begin{aligned}\frac{d^2g_\circ}{dt^{*2}}+\left(ag_\circ^2-b\right)\frac{dg_\circ}{dt^*}&=0\\ \frac{d^2g_1}{dt^{*2}}+\left(ag_\circ^2-b\right)\frac{dg_1}{dt*}+2ag_\circ g_1\frac{dg_\circ}{dt^*}&=0\end{aligned}\right\} \tag{2.21}$$

if $0(\beta_1)>0(\varepsilon)$. If, however, $\beta_1=\varepsilon$ then we have

$$\frac{d^2g_1}{dt^{*2}}+\left(ag_\circ^2-b\right)\frac{dg_1}{dt^*}+2ag_\circ g_1\frac{dg_\circ}{dt*}+g_\circ+cg_\circ^3=0$$

Integration of (2.21) yields

$$\frac{dg_\circ}{dt^*}+\frac{ag_\circ^3}{3}-bg_\circ=k_3\ \text{(constant)} \tag{2.22}$$

$$\frac{dg_1}{dt^*}+\left(ag_\circ^2-b\right)g_1=k_4\ \text{(constant)} \tag{1.23}$$

In the matching of the inner expansion to the outer expansion, an intermediate class of limits of the form

$$\left(\varepsilon\to 0, t_\eta=\frac{t-\delta(\varepsilon)}{\eta}\ \text{fixed}\right), \frac{\eta}{\varepsilon}\to\infty, \eta\to 0$$

is considered, so that $t=\eta t_\eta+\delta(\varepsilon)\to 0$, $(t_\eta<0)$

$$t*=\frac{\eta t_n+\delta(\varepsilon)-\delta(\varepsilon)}{\varepsilon}=\eta t_n\to-\infty$$

since $x_\circ\to\sqrt{b/a}$ as $t\to 0$, for matching to be possible to the lowest order, we must also have that $g_\circ\to\sqrt{b/a}$ as $t*\to-\infty$ and $k_3=-\frac{2b}{3}\sqrt{b/a}$, and (1.22) becomes

$$\frac{dg_0}{dt*}=-\frac{a}{3}\left(\sqrt{\frac{b-g_0}{a}}\right)\left(g_0+2\sqrt{\frac{b}{a}}\right) \tag{2.24}$$

For matching to the other branch of the outer solution CD (Figure 1.2), we expect that t* → + ∞ and from (1.24), it follows that $g_\circ\to-2\sqrt{b/a}$. Hence, for matching to the first order, $x_\circ\to-2\sqrt{b/a}$ as $t\to 0^+$. These then give the first approximation to the size of the discontinuity; x_0 goes from $\sqrt{b/a}$ to $-2\sqrt{b/a}$ and hence the first approximation to the period. We now integrate (2.24) to obtain

$$\frac{\frac{1}{a}}{\sqrt{\frac{b}{a}}-g_0}\frac{1}{3b}\log\left(\sqrt{\frac{b}{a}}-g_0\right)+\frac{1}{3b}\log\left(g_0+2\sqrt{\frac{b}{a}}\right)=-t* \qquad (2.25)$$

and expanding as $t*\to-\infty$

$$g_0=\sqrt{\frac{b}{a}}+\frac{\gamma_1}{t*}+\gamma_2\frac{\log(-t*)}{t*^2} \qquad (2.26)$$

and so (2.25) becomes, on substitution of (2.26)

$$-\frac{1}{a\sqrt{b/a}}\frac{t*}{\gamma_1}+\frac{\gamma_2\log(-t*)}{a\sqrt{b/a}\gamma_1}-\frac{\gamma_2^2}{\gamma_1}\frac{1}{a\sqrt{b/a}}\frac{\log^2(-t*)}{t*}+\frac{\log 3\sqrt{b/a}}{3b}+\frac{1}{9b\sqrt{b/a}}\frac{\log(-t*)}{t^{*2}}$$

$$-\frac{\log\gamma_1}{3b}+\frac{\log(t-t*)}{3b}-\frac{\gamma_2}{3b\gamma_1}\frac{\log(-t*)}{t}+\frac{\gamma_2}{6b\gamma_1^2}\frac{\log^2(-t*)}{t^{*2}}+\ldots=-t*$$

Equating powers of t* we obtain

$$\gamma_1=\frac{1}{a}\sqrt{\frac{a}{b}},\ \gamma_2=-\frac{1}{3b^2}\sqrt{\frac{b}{a}}$$

so that (2.26) becomes

$$g_\circ=\sqrt{\frac{b}{a}}+=\frac{1}{\sqrt{ab}}\frac{1}{t*}-\frac{1}{3b\sqrt{ab}}\frac{\log(-t*)}{t^{*2}}+\ldots \qquad (2.27)$$

with general solution given by

$$\frac{dg_1}{dt*}-\left(b-ag_\circ^2\right)g_1=k_4 \qquad (2.28)$$

where h_1 is a constant and g_{1p} is any particular solution. We can find the behaviour of the particular solution as t* → - ∞. Thus, $\frac{dg_{1p}}{dt*}-\left(-\frac{2}{t*}+\frac{2}{3b}\frac{\log(-t*)}{t^{*2}}-\frac{1}{bt^{*2}}+\ldots\right)g_{1p}=k_4$. Assuming the particular solution in the form $g_{1p}=A_1t+B_1\log(-t*)+C_1\cdots$. The coefficient A_1, B_1 and C_1 are readily determined to give

$$g_{1p}=\frac{k_2}{3}\left\{t*+\frac{1}{3b}\log(-t*)-\frac{1}{3b}+\frac{\log^2(-t*)}{t^{*2}}\right]\Bigg\},t*\to-\infty \qquad (2.29)$$

if $k_4\neq0$, the particular solution dominates as t* → - ∞, and so the general solution is given by

$$g_{1p}=\frac{K_4}{3}\left\{t*+\frac{1}{3b}\log(-t*)-\frac{1}{3b}+0\left(\frac{\log^2(-t*)}{t^{*2}}\right)\right\}+h_1\left\{-\frac{1}{\sqrt{ab}}\frac{1}{t^{*2}}+0\left(\frac{\log(-t*)}{t^{*2}}\right)\right\}t*\to-\infty \qquad (2.30)$$

$t*\to-\infty$

The inner expansion therefore becomes

$$x(t;\varepsilon)=\sqrt{\frac{b}{a}+\frac{1}{\sqrt{ab}}\frac{1}{t*}}-\frac{1}{3b\sqrt{ab}}=\frac{\log(-t*)}{t^{*2}}+\ldots+\beta_1(\varepsilon)\left\{\frac{k_4t*}{3}+\frac{k_4}{6b}\log(-t*)-\frac{k_4}{9b}+\ldots-\frac{h_11}{\sqrt{abt^{*2}}}+\ldots\right\}+\ldots$$

(2.31)

In order to see the possibility of matching, we write both the inner and outer expansions in terms of intermediate variables,

$$t=\eta t_n+\delta(\varepsilon)\to0,\eta/\varepsilon\to\infty,\eta\to0,\ t*=\frac{\eta t_n}{\varepsilon}\to-\infty,(t_\eta<0).$$

We can therefore write the *outer* and *inner* expansions respectively as

$$x(t_i\varepsilon)=\sqrt{\frac{b}{a}}1+\left\{1+\sqrt{\frac{a+bc}{ab}}\sqrt{-\mu t_n-\delta}+\frac{a+5bc}{6ab}\left(-\eta t_n-\delta\right)+\ldots\right\}$$

$$+\varepsilon\left\{1+\sqrt{\frac{1}{ab(-\eta t_\eta-\delta)}}+\frac{a+3bc}{8ab\sqrt{a+bc}}-\frac{1}{\sqrt{\eta t_\eta-\delta}}\log\left(-\eta t_n-\delta\right)+\ldots\right\}+\ldots \qquad (2.32)$$

and

$$x(t_i\varepsilon)=\sqrt{\frac{b}{a}}+\frac{1}{\sqrt{ab}}\frac{\varepsilon}{\eta t_\eta}-\frac{\varepsilon^2}{3\sqrt{ab}}\frac{\log(-\eta t/\varepsilon)}{(\eta t_\eta)}+\ldots+\beta_1(\varepsilon)\left\{\frac{k_4\eta t_\eta}{3\varepsilon}+\frac{k_4}{9b}\log(-\eta t_\eta/\varepsilon)-\frac{k_4}{9b}+\ldots-\frac{h_1\varepsilon^2}{\sqrt{ab(\eta t_\eta)}}\right\}+\ldots$$

(2.33)

It is clear from (2.32) and (2.33) that no term in (2.32) can match the term $0(1/\eta t_\eta)$ in (2.33). This failure to match therefore implies the existence of a distinguished limit and a transition expansion such as would match the outer solution as $t \to 0^-$ and the inner as $t^* \to -\infty$.

3.0 Asymptotic expansions of the transition layer and matching

The failure of the outer and inner expansions to match gives rise to the introduction of a continuum of intermediate limits, lying between the two expansions, called the transition expansion. We shall develop this transition expansion and match it to the outer and inner expansions. Consider the limit process

$$\varepsilon \to 0, \text{ŧ} = \frac{t-\rho(\varepsilon)}{v(\varepsilon)} \text{ fixed, } 1 >> v(\varepsilon) >> \varepsilon \qquad (3.1)$$

The first terms in each of the expansions (3.2) and (1.33) are already matched, thus the transition expansion assumes the form $x(t;\varepsilon)=\sqrt{b/a}+\sigma_1(\varepsilon)f_1(\text{ŧ})+\sigma_2(\varepsilon)f_2(\text{ŧ})+\ldots\ldots$ (3.2)

and the equation (1.1) becomes

$$\frac{\varepsilon}{v^2}\left(\sigma_1\frac{d^2}{d\text{ŧ}}\frac{f_1}{2}+\sigma_2\frac{d^2f_2}{d^2}+\ldots\right)+\left(2a\sqrt{\tfrac{b}{a}}\sigma_1f_1+2a\sqrt{\tfrac{b}{a}}\sigma_2f_2+a\sigma_1^2f_1^2+\ldots\right)\frac{1}{v}\left[\sigma_1\frac{df_1}{d\text{ŧ}}+\right.$$

$$\left.+\sigma_2\frac{df_2}{d\text{ŧ}}+\ldots\right]+\sqrt{\frac{b}{a}+\sigma_1f_1+\sigma_2f_2+\cdots+c\left\{\frac{b}{a}\sqrt{\frac{b}{a}}+\frac{3b}{a}(\sigma_1f_1+\sigma_2f_2)+\right.}$$

$$\left.+3\sqrt{\frac{b}{a}}\sigma_1^2f_1^2+\sigma_1^3f_1^3+6\sqrt{\frac{b}{a}}\sigma_1\sigma_2f_1f_2+\ldots\right\}=0 \qquad (3.3)$$

The orders of the terms associated with derivatives of each order are $\varepsilon\frac{\sigma_1}{v^2}\leftrightarrow\frac{\sigma_1^2}{v}\leftrightarrow 1$. The intermediate limit is that in which all these orders are equal. Thus $\sigma_1=\varepsilon^{1/3}$, $v=\varepsilon^{2/3}$ and, for the equation for f_2 to contain forcing terms, we must have that $\sigma_2=\varepsilon^{2/3}$. The transition equation (2.2) becomes

$$x(t;\varepsilon)=\sqrt{\frac{b}{a}}+\varepsilon^{1/3}f_1(\text{ŧ})+\varepsilon^{2/3}f_2(\text{ŧ})+\ldots\ldots, \text{ŧ}=\frac{t-\rho(\varepsilon)}{\varepsilon^{2/3}} \qquad (3.4)$$

From (3.3) we have the equations

$$\frac{d^2f_1}{d\text{ŧ}^2}+2a\sqrt{\frac{b}{a}}f_1\frac{df_1}{d\text{ŧ}}+\sqrt{\frac{b}{a}}\left(1+\tfrac{bc}{a}\right)=0 \qquad (3.5)$$

$$\frac{d^2f_1}{d\text{ŧ}^2}+2a\sqrt{\frac{b}{a}}\frac{d(f_1f_2)}{d\text{ŧ}}+af_1^2\frac{df_1}{dt}+\left(1+\tfrac{3bc}{a}\right)f_1=0 \qquad (3.6)$$

Integrating (3.5) with $f_1 \quad \frac{V'(\text{ŧ})}{a\sqrt{b/a}V(\text{ŧ})}=\frac{1}{a\sqrt{b/a}}\frac{d}{dt}\log(V(\text{ŧ}))$, we obtain

$$\frac{d^2V}{d\text{ŧ}^2}+\frac{b(a+bc)}{a}\text{ŧ}V=0 \qquad (3.7)$$

This is a form of the Airy equation, the general solution of which can be written in terms of the modified Bessel functions of the first and third kind, $I_{1/3}$ and $K_{1/3}$ respectively. If we let $\tau=-\alpha\text{ŧ}$ where $\alpha^3=(b/a)(a+bc)$, then (3.7) becomes

$$\frac{d^2V}{d\tau^2}-\tau V=0 \qquad (3.8)$$

Solving this equation (3.8) for V, we obtain $V = M\tau^{1/2}K_{1/3}\left(\tfrac{2}{3}\tau^{3/2}\right) + N\tau^{1/2}I_{1/3}\left(\tfrac{2}{3}\tau^{3/2}\right)$ (3.9)

where M, N are constants. At $t \to -\infty, K_{1/3}$ and $I_{1/3}$

$$K_{1/3}\left(\tfrac{2}{3}\sqrt{(b/a)(a+bc)}(-t)^{3/2}\right) \approx \tfrac{1}{2}\sqrt{3\pi}\left[(b/a)(a+bc)\right]^{-1/4}(-t)^{-3/4}e^{-2/3}\sqrt{(b/a)(a+be)}(-t)^{3/2}$$
$$\left\{1-\frac{5}{48\sqrt{(b/a)(a+bc)}}\frac{1}{(-t)^{3/2}}+\cdots\right\} \qquad (3.10)$$

and

$$I_{1/3}\left(\tfrac{2}{3}\sqrt{(b/a)(a+bc)}(-t)^{3/2}\right) \approx \tfrac{1}{2}\sqrt{3\pi}\left[(b/a)(a+bc)\right]^{-1/4}(-t)^{-3/4}e^{-2/3}\sqrt{(b/a)(a+bc)}(-t)^{3/2}$$
$$\left\{1-\frac{5}{48\sqrt{(b/a)(a+bc)}}\frac{1}{(-t)^{3/2}}+\cdots\right\} \qquad (3.11)$$

As $t \to -\infty$, $f_1 = \pm(1/a)\sqrt{a+bc}\,\sqrt{-t}+\ldots$

For matching to be achieved with the expansion of x_o as $t \to 0^-$, we must take the positive sign. From the asymptotic behaviour of $K_{1/3}$ and $I_{1/3}$, we have that

$$V(t) = \frac{M\sqrt{3\kappa}}{2}R^{\frac{-1}{12}}(-t)^{\frac{-2}{3}\sqrt{R}(t)^{\frac{3}{2}}}\left\{1-\frac{5}{48\sqrt{R}}\frac{1}{(-1)^{\frac{3}{2}}}+\cdots\right\}+\frac{N}{2}\sqrt{\frac{3}{\kappa}}R^{\frac{-1}{12}}(-t)^{\frac{-1}{4}}e^{\frac{2}{3}\sqrt{R}(-t)^{\frac{3}{2}}}\left\{1-\frac{5}{48\sqrt{R}}\frac{1}{(-1)^{\frac{3}{2}}}+\cdots\right\} \qquad (3.12)$$

where $R = (b/a)(a + bc)$, so that

$$\log V(t) = \begin{cases} \frac{2}{3}\sqrt{R}(-t)^{3_2}+\cdots, \ if \ \ N \neq 0 \\ \frac{2}{3}\sqrt{R}(-t)^{\frac{3}{2}}+\cdots, \ if \ \ N = 0 \end{cases}$$

From the transformation, $f_1 = (1/a)d/dt(\log V(t))$ and has proper behaviour only if $N = 0$. Equation (3.12) then

becomes $$V(t) = \frac{M\sqrt{3\kappa}}{2}R^{\frac{-1}{12}}(-t)^{\frac{-2}{3}\sqrt{R}(t)^{\frac{3}{2}}}\left\{1-\frac{5}{48\sqrt{R}}\frac{1}{(-1)^{\frac{3}{2}}}+\cdots\right\} \qquad (3.13)$$

Similarly we can construct the solution f_2; and obtain

$$f_2 = \frac{D_2}{V^2}-\left(\frac{a\sqrt{abC_2}}{b(a+bc)}-\frac{\log V}{3b}\right)\frac{df_1}{dt}-\frac{\sqrt{ab}}{6b}f_1^{\,2}-\frac{2(a+4bc)}{3a\sqrt{ab}V^2}\int_{-\infty}^{t}V^2\log V d\lambda \qquad (3.14)$$

We need to determine the asymptotic behaviour of $\int_{-\infty}^{t}V^2\log V d\lambda$. Using (3.13) and letting $\alpha^3 = (b/a)(a + bc)$, we

have $V^2(t) = \frac{M^2\pi}{4}(-\alpha t)^{-\frac{1}{2}}e^{-\frac{4}{3}(-\alpha t)^{\frac{3}{2}}}\left\{1-\frac{5}{24}(-\alpha t)^{-\frac{3}{2}}+\cdots\right\}$ so that $\int_{-\infty}^{t}V^2\log V d\lambda =$

$$\frac{3M^2\pi}{4}(-\alpha t)^{1}e^{-\frac{4}{3}(-\alpha t)^{\frac{3}{2}}}\left\{1-\frac{5}{3\alpha}(-\alpha t)^{-\frac{3}{2}}-\frac{\log(-\alpha t)}{8\alpha}+\frac{1}{2\alpha}\log\frac{M\sqrt{3\pi}}{2}+0\left((-t)^{\frac{-3}{2}}\right)\right\}$$

Also, from (3.13), f_1 is given by

$$f_1 = \frac{V'}{\sqrt{ab}V} = \frac{1}{\sqrt{ab}}\left\{\sqrt{\left(\frac{b}{a}\right)(a+bc)-t}\,\frac{1}{4(-t)}-\frac{5(-t)}{32\sqrt{\left(\frac{b}{a}\right)(a+bc)}}\right\}+0\left(\frac{1}{t^4}\right) \qquad (3.15)$$

The terms of (3.15) can therefore be determined and hence

$$f_2 \approx \frac{a+5bc}{6a\sqrt{ab}}(-t)+\frac{a+3bc}{8ab\sqrt{ab+c}}\log\frac{(-t)}{\sqrt{-t}}+\frac{1}{\sqrt{-t}}\left\{\frac{C_2}{2\sqrt{\left(\frac{b}{a}\right)(a+bc)}}-\frac{a+bc}{12ab\sqrt{ab+bc}}-\frac{a+3bc}{2ab\sqrt{ab+c}}\log\frac{M\sqrt{3\pi}}{2}\right.$$

$$\left.+\frac{a+3bc}{24ab\sqrt{ab+c}}\log\left(\frac{b}{a}\right)(a+bc)+\cdots\right\} \qquad (3.16)$$

The *transition* expansion therefore can be written as

$$X(t,\varepsilon)=\sqrt{\frac{b}{a}}+\varepsilon^1\left\{\frac{\sqrt{a+5bc}}{a}(-t)+\frac{1}{4\sqrt{ab(-t)}}+\cdots\right\}+\varepsilon^2\left\{\frac{(a+5bc)(-t)}{6a\sqrt{ab}}+\frac{a+3bc}{8ab\sqrt{ab+bc}}\log\frac{(-t)}{\sqrt{-t}}\right.$$

$$\left.+\frac{1}{\sqrt{-t}}\left[\frac{C_2}{2\sqrt{\left(\sqrt{\frac{b}{a}}\right)(a+bc)}}-\frac{a+bc}{12ab\sqrt{a+bc}}-\frac{a+3bc}{2ab\sqrt{a+bc}}\log\frac{M\sqrt{3\pi}}{2}+\frac{a+3bc}{24ab\sqrt{a+bc}}\log\frac{\frac{b}{a}(a+bc)}{2}\right]+\cdots\right\}+$$

(3.17)

We now consider matching to the branch AB of the *outer* expansion. The intermediate limit is defined by Hence the *outer* expansion can be written in the form

$$\varepsilon\to 0,\quad t_\eta=\frac{t-\rho(\varepsilon)}{\eta(\varepsilon)}\ \text{ fixed},\ \varepsilon^{\frac{2}{3}}\prec\prec\eta\prec\prec 1 \qquad (3.18a)$$

that is, $$t=\eta\,t_\eta+\rho(\varepsilon)\to 0,\quad t=\frac{\eta\,t_\eta}{\varepsilon^{\frac{2}{3}}}\to-\infty \qquad (3.18b)$$

$$X(t,\varepsilon)=\sqrt{\frac{b}{a}}+\varepsilon^1\frac{a+bc\sqrt{-\eta t_\eta-\rho}}{a}+\frac{a+5bc(\eta t_\eta-\rho)-}{6a\sqrt{ab}}+\cdots+\varepsilon^2\left\{\frac{1}{4\sqrt{ab}(\eta t_\eta-\rho)}+\frac{a+3bc}{8ab\sqrt{ab+bc}}\log\frac{(\eta t_\eta-\rho)}{\sqrt{-\eta t_\eta-\rho}}\right.$$

$$\left.+\frac{1}{2a\sqrt{(-\eta t_\eta-\rho)(a+bc)}}\left[(a+bc)K_2-\frac{a-bc}{6b}+\frac{a+3bc}{2b}\log 2\sqrt{\left(\frac{b}{a}(a+bc)\right)}+\frac{a+bc}{b}\log\sqrt{\frac{b}{a}}-c\log\left(\frac{(a+bc)}{a}\right)\right]\right\}+\cdots$$

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(3.19)

For ρ << η, we obtain

$$X(t,\varepsilon)=\sqrt{\frac{b}{a}}+\frac{\sqrt{a+bc}}{a}+\left(-\eta t_\eta-\frac{\rho}{2\sqrt{-\eta t_\eta}}+\cdots\right)+\frac{a+5bc}{6a\sqrt{ab}}(-\eta t_\eta-\rho)+\varepsilon\left\{\frac{1}{4\sqrt{ab}(-\eta t_\eta)}+\frac{\rho}{4\sqrt{ab}(-\eta t\eta)^2}+\cdots\right.$$

−

$$+\frac{a+3bc}{8ab\sqrt{a+bc}}+\left[\frac{\log(-\eta t_\eta)}{\sqrt{-\eta t_\eta}}\frac{\rho}{(-\eta t_\eta)^{\frac{3}{2}}}+\frac{\rho}{2(-\eta t_\eta)^{\frac{3}{2}}}\log(-\eta t_\eta)+\cdots\right]+\frac{1}{2a\sqrt{(-\eta t_\eta)(a+bc)}}\left[(a+bc)K_2-\frac{a-bc}{6b}\right.$$

$$\left.\left.+\frac{a+3bc}{2b}\log 2+\frac{3a+5bc}{4b}\log\left(\frac{b}{a}\right)+\frac{a-bc}{4b}\log(a+bc)+c\log a\right]\right\}+\cdots \qquad (3.20)$$

The *transition* expansion obtained using the intermediate variables is

$$X(t,\varepsilon)=\sqrt{\frac{b}{a}}+\varepsilon^{1}\left\{\frac{a+bc\sqrt{-\eta t_{\eta}}}{\varepsilon^{\frac{2}{3}}}+\frac{\varepsilon^{\frac{2}{3}}}{4\sqrt{ab}(-\eta t_{\eta})}+\cdots\right\}+\varepsilon^{\frac{2}{3}}\left\{\frac{a+3bc}{8ab\sqrt{a+bc}}\frac{\varepsilon^{\frac{3}{2}}}{\sqrt{-\eta t_{\eta}}}\left[\log(-\eta t_{\eta})-\frac{2}{3}\log\varepsilon\right]\right.$$

$$+\frac{a+5bc}{6a\sqrt{ab}}\frac{(-\eta t_{\eta})}{\varepsilon^{\frac{2}{3}}}+\frac{\varepsilon^{\frac{2}{3}}}{\sqrt{-\eta t_{\eta}}}\left[\frac{C_2}{2\sqrt{\frac{b}{a}(a+bc)}}-+\frac{a+bc}{12ab\sqrt{a+bc}}-\frac{a+3bc}{2ab\sqrt{a+bc}}\log\frac{M\sqrt{3\pi}}{2}\right.$$

$$\left.\left.+\frac{a+3bc}{24ab\sqrt{a+bc}}\log\frac{b}{a}(a+bc)+\cdots\right]\right\} \qquad (3.21)$$

For matching we compare (3.20) and (3.21). The terms $0\left(\sqrt{-\eta t_{\eta}}\right)$, $0(-\eta t_{\eta})$, $0\left(\frac{\log(-\eta t_{\eta})}{\sqrt{-\eta t_{\eta}}}\right)$ all match exactly. We therefore consider the terms $0\left(\frac{1}{\sqrt{-\eta t_{\eta}}}\right)$. Equations (3.21) and (2.2) will match if we choose ρ and K_2 as

$$\rho(\varepsilon)=\frac{a+3bc}{6b}\varepsilon\log\varepsilon$$

$$K_2=\frac{\sqrt{ab}C_2}{b(a+bc)}-\frac{a+3bc}{b(a+bc)}\frac{\log M\sqrt{3\pi}}{2}-\frac{a+3bc}{2b(a+bc)}\log 2-\frac{1}{b(a+bc)}\left\{\frac{bc}{3}+\frac{a+3bc}{6}\log\left(\frac{b}{a}\right)(a+bc)\right.$$
$$\left.+\frac{a+bc}{2}\log\left(\frac{b}{a}\right)-bc\log\left[\frac{(a+bc)}{a}\right]\right\} \qquad (3.22)$$

The terms omitted vanish more rapidly than those matched. Next, we match the *transition* expansion (3.17) to the inner expansion as $t^* \to -\infty$. At the first zero of the Airy function $V(t)$, the function V vanishes and therefore f_1 becomes infinite there. From the behaviour of f_1 the matching will take place as $t \to t_0$ which is the first zero of $V(t)$. We therefore need the behaviour of f_1, f_2 as $t \to t_0$. $V(t)$ has a simple zero at $t = t_0$ and so from (3.7) it has an expansion near t_0 of the form

$$V(t)=M\left\{-K(t,-t_0)\frac{K\left(\frac{b}{a}\right)(a+bc)}{6}t_0(t-t_0)^3+0(t-t_0)\right\} \qquad (3.23)$$

where K is a constant. $V(t)=M\left\{-K+\frac{K\left(\frac{b}{a}\right)(a+bc)}{6}t_0(t-t_0)^2+0(t-t_0)+\cdots\right\}$. Thus as $t \to t_0$

$$f_1(t)=\frac{1}{\sqrt{ab}}\left\{\frac{1}{t-t_0}-\frac{b(a+bc)}{3a}t_0(t-t_0)^2+0(t-t_0)\right\} \qquad (3.24)$$

The integral $\int_{-\infty}^{t}V^2\log V d\lambda$ in (2.14) approaches a finite value as $t \to t_0$ and therefore the dominant term in f_2 comes from $\frac{1}{3b}\log V\frac{df_1}{dt}$. Thus

$$f_2=\frac{1}{3b\sqrt{ab}}\frac{\log(t-t_0)}{(t_0-t)^2}+0\left(\frac{1}{(t_0-t)^2}\right) \qquad (3.25)$$

To express the intermediate limit in this case, we first write t^* in terms of t. From (3.18)

$$t^* = \frac{t-\delta(\varepsilon)}{\varepsilon} = \frac{t+\varepsilon^{-\frac{2}{3}}\rho - \varepsilon^{-\frac{2}{3}}\delta(\varepsilon)}{\varepsilon^{-2\frac{}{3}}}$$

and if we let $\delta(\varepsilon) = \varepsilon^{-\frac{2}{3}}\{t_0 + \gamma(\varepsilon)\}$, we obtain $t^* = \frac{t - t_0\gamma(\varepsilon) + \varepsilon^{-\frac{2}{3}}\rho}{\varepsilon} = \frac{t + t_0 - \sigma(\varepsilon)}{\varepsilon^{-2\frac{}{3}}}$, where $\sigma(\varepsilon) = \gamma(\varepsilon) - \varepsilon^{-\frac{2}{3}}\rho = \gamma(\varepsilon) - \frac{a+3bc}{6b}\varepsilon^{\frac{2}{3}}\log\varepsilon$. We can therefore express the intermediate class of limits in (3.18) as

$$\varepsilon \to 0,\ t_\eta = \frac{t - t_0 - \sigma(\varepsilon)}{\eta},\ \text{fixed},\ \varepsilon^{\frac{2}{3}} \prec\prec \eta \prec\prec 1$$

so that $$t_\eta = \frac{\varepsilon^{\frac{2}{3}} t^*}{\eta} \Rightarrow t^* = \frac{\eta t_\eta}{\varepsilon^{\frac{2}{3}}} \to -\infty,\ (t_\eta \prec 0),\ t - t_0 \to \eta t_\eta + \sigma(\varepsilon)$$

Using the expression for f_1 and f_2 in (3.24) and (3.25) we write the transition expansion in intermediate variables as $\sigma \prec\prec \eta$, we obtain

$$X(t,\varepsilon) = \sqrt{\frac{b}{a}} + \varepsilon^{\frac{2}{3}}\left\{\frac{1}{\sqrt{ab}(\eta t_\eta + \sigma)} - \frac{b}{3a\sqrt{ab}}(a+bc)t_0(\eta t_\eta + \sigma) + \cdots\right\} + \varepsilon^{\frac{2}{3}}\left\{-\frac{\log(\eta t_\eta + \sigma)}{3b\sqrt{ab}(\eta t_\eta + \sigma)} + \cdots\right\} +$$

and, if

$$X(t,\varepsilon) = \sqrt{\frac{b}{a}} + \varepsilon^{\frac{2}{3}}\left\{\frac{1}{\sqrt{ab}(-\eta t_\eta)} - \frac{\varpi}{(-\eta t_\eta)} + \frac{b(a+bc)t_0(-\eta t_\eta)}{3a\sqrt{ab}} + \cdots\right\} - \frac{\varepsilon^{\frac{2}{3}}\log(-\eta t_\eta)}{3b\sqrt{ab}(-\eta t_\eta)^2} + \cdots \quad (3.26)$$

and from (2.31). the inner expansion becomes

$$X(t,\varepsilon) = \sqrt{\frac{b}{a}} - \frac{\varepsilon^{\frac{2}{3}}}{\sqrt{ab}(-\eta t_\eta)} + \frac{\varepsilon^{\frac{2}{3}}\log(-\eta t_\eta)}{3b\sqrt{ab}(-\eta t_\eta)} + \frac{\varepsilon^{\frac{2}{3}}\log\varepsilon}{9a\sqrt{ab}(-\eta t_\eta)^2} + \cdots$$
$$+\beta_1(\varepsilon)\left\{\frac{k_3}{3}\left(\frac{(-\eta t_\eta)}{\varepsilon^{\frac{2}{3}}}\right) + \frac{k_3\log(-\eta t_\eta)}{9b} + \frac{k_3\log\varepsilon}{27b(-\eta t_\eta)} + \cdots + \frac{k_1\varepsilon^{\frac{2}{3}}(-\eta t_\eta)}{\sqrt{ab}(-\eta t_\eta)^2} + \cdots\right\} + \cdots \quad (3.27a)$$

The terms in (3.26) and (3.27) match exactly, if we make

$$\beta_1 = \varepsilon^{\frac{2}{3}},\ k_3 = \frac{b(a+bc)}{\sqrt{ab}}t_0,\ \sigma = \frac{\varepsilon^{\frac{2}{3}}\log\varepsilon}{9b\sqrt{ab}} \quad (3.27b)$$

Thus, $\gamma(\varepsilon) = \sigma + \varepsilon^{-\frac{2}{3}}\rho = \frac{\varepsilon^{-\frac{2}{3}}\log}{18b}\left(3(a+3bc) - \frac{2}{\sqrt{ab}}\right)$. $\delta(\varepsilon) = \varepsilon^{-\frac{2}{3}}(t_0 + \gamma(\varepsilon)) = \varepsilon^{-\frac{2}{3}}t_0 + \frac{\varepsilon\log\varepsilon}{18b}\left\{3(a+3bc) - \frac{2}{\sqrt{ab}}\right\}$

and $$t^* = \left(\frac{1}{\varepsilon}\right)\left\{t - \varepsilon^{-\frac{2}{3}}t_0 - \frac{\varepsilon\log\varepsilon}{18b}\left\{3(a+3bc) - \frac{2}{\sqrt{ab}}\right\}\right\} \quad (3.28)$$

We finally close the cycle by matching the *inner* expansion as t* → ∞ to the branch CD of the outer expansion as t → 0^{+}, $x_0 \to -2\sqrt{\frac{b}{a}}$. From (2.7) CD can be expressed as both reflection and translation of AB, that is

$$b\log\left(-\frac{x_0}{\sqrt{ab}}\right) - \frac{a+bc}{2c}\log\left(\frac{1+cx_0^2}{1+\frac{b}{c}}\right) = 1 \quad (3.29)$$

where $t^* = t - ½T(\varepsilon)$, $T(\varepsilon)$ is the period on the scale t. Expanding (3.29) about $x_0 = -2\sqrt{\frac{b}{a}}$ x_0 and letting $y_0 = \frac{x_0}{\sqrt{\frac{b}{a}}}$

we have

$$b\log 2+\frac{3ab}{2\sqrt{\frac{b}{a}}(a+4bc)}\xi+\left[\frac{a}{8}-\frac{a(a+bc)}{2(a+4bc)}-\frac{4(a+bc)abc}{(a+4bc)}\right]\xi^2+\cdots+\frac{a=bc}{2c}\log\frac{a+4bc}{a+bc}=t \tag{3.30}$$

where $\xi = x_0 + 2\sqrt{\frac{b}{a}}$. To the lowest order in ξ, we have

$$\xi=\frac{2\sqrt{\frac{b}{a}}(a+4bc)}{3ab}\left\{t^*-b\log 2+\frac{a+bc}{2c}\log\frac{a+4bc}{a+bc}\right\}+\cdots \tag{3.31}$$

$$\Rightarrow X_0(t^*)=-2\sqrt{\frac{b}{a}}+2\sqrt{\frac{b}{a}}\frac{(a+4bc)}{3ab}\left\{t^*-b\log 2+\frac{a+bc}{2c}\log\frac{a+4bc}{a+bc}\right\}+\cdots$$

From (3.24) we have that as $t^* \to \infty$, $g_0(t^*)=-2\sqrt{\frac{b}{a}}+0\left(e^{-3t^*}\right)$ and the particular solution g_{1p} of (1.28) satisfies the equation $\frac{dg_{1p}}{dt^*}+\left\{3b+0\left(e^{-3t^*}\right)\right\}g_{1p}=k_3=-\frac{b(a+bc)}{a\sqrt{ab}}t_0\log\frac{a+4bc}{a+bc}$ from which we have

$$g_{1p}=-\frac{a+bc}{3a\sqrt{ab}}t_0+\cdots \text{ as } t^*\to\infty, \tag{3.32}$$

Using (3.28) the relationship between $t^*$and t^+ can be obtained as

$$t^*=\left(\frac{1}{\varepsilon}\right)\left\{t+\tfrac{1}{2}T(\varepsilon)-\varepsilon^{-\frac{2}{3}}t_0-\frac{\varepsilon\log\varepsilon}{18b}\left\{3(a+3bc)-\frac{2}{\sqrt{ab}}\right\}\right\} \tag{3.33}$$

Thus the intermediate limit for this matching has $\varepsilon \to \infty$, t_η fixed

where $t_\eta=\frac{1}{\eta(\varepsilon)}t^*$, $\varepsilon \ll \eta \ll 1$, and $t^*=\frac{\eta t_\eta}{\varepsilon}\to\infty$, $t^+=-\tfrac{1}{2}T(\varepsilon)+\varepsilon^{\frac{2}{3}}t_0-\frac{\varepsilon\log\varepsilon}{18b}\left\{3(a+3bc)-\frac{2}{\sqrt{ab}}\right\}+\eta t_\eta$

We now write the *outer* expansion in t_η from (3.31) as

$$X(t,\varepsilon)=-2\sqrt{\frac{b}{a}}+\left(\frac{2(a+4bc)}{3a\sqrt{ab}}\right)\left\{\frac{1}{2}T(\varepsilon)+\varepsilon^{\frac{2}{3}}t_0-(\varepsilon\log\varepsilon)\left(\frac{3(a+3bc)-2\sqrt{\frac{b}{a}}}{18ab}\right)+\eta t_\eta-b\log 2+\frac{a+bc}{2c}\log\left(\frac{a+4bc}{a+bc}\right)\right\}+\cdots+0(\varepsilon) \tag{3.34}$$

The *inner* expansion is similarly $X(t,\varepsilon)=-2\sqrt{\frac{b}{a}}+\varepsilon^{\frac{2}{3}}\left(\frac{-a+bc}{3a\sqrt{ab}}t_0\right)+\cdots+\beta_2(\varepsilon)g_2+\cdots\left(\beta_2\prec\prec\varepsilon^{\frac{2}{3}}\right)$ (3.35)

The period, T(ε) is therefore obtained when we compare the two equations (3.34) and (3.35). We obtain

$$T(\varepsilon)=\frac{a+bc}{c}\log\left(\frac{a+4bc}{a+bc}\right)-2\log 2+\frac{3(a+3bc)}{a+4bc}\varepsilon^{\frac{2}{3}}t_0+0(\varepsilon\log\varepsilon) \tag{3.36}$$

Thus the uniformly valid expansion of the given equation (3.1) obtained by adding the *outer* and *inner* expansions and then subtracting the common part, is given by

$$X(t,\varepsilon)=\sqrt{\frac{b}{a}}+\frac{\sqrt{a+bc}}{a}\sqrt{-t}+\frac{a+5bc}{6a\sqrt{ab}}\sqrt{-t}+\varepsilon\left\{\frac{1}{4\sqrt{abt}(-t)}+\frac{a+3bc\log(-t)}{8ab\sqrt{a+bc}\sqrt{-t}}+\frac{1}{\sqrt{-t}}\left[\frac{a+bc}{2a}k_2-\frac{a-bc}{12ab}\right.\right.$$

$$\left.\left.+\frac{a+3bc}{4ab}\log 2\sqrt{\frac{b}{a}}(a+bc)+\frac{a+bc}{2ab\sqrt{a+bc}}log\sqrt{\frac{b}{a}}+\frac{c}{2a}\log\frac{a+bc}{a}+\cdots\right]+\frac{1}{\sqrt{a+bc}}\right\}+\cdots+\frac{1}{\sqrt{abt^*}}-\frac{1}{3b\sqrt{ab}}\frac{\log(-t^*)}{t^{*2}}+\cdots \tag{3.37}$$

$$+\left(\frac{\varepsilon^{\frac{2}{3}}k_3}{3}\right)\left\{t^*+\left(\frac{1}{b}\right)\log(-t^*)\right]+\frac{1}{3b}\right\}+\cdots$$

where k_2 and k_3 are defined respectively in (3.22) and (3.27b) and t* is defined in (3.28). The equation (3.37) therefore gives the mode shape of the *relaxation oscillation* of (2.1).

4.0 Conclusion

We have obtained the following asymptotic expression

$$T(\varepsilon)=\frac{a+bc}{c}\log\left(\frac{a+4bc}{a+bc}\right)-2b\log 2+\frac{3(a+3bc)}{a+4bc}\varepsilon^{\frac{2}{3}}t_0+0(\varepsilon\log\varepsilon) \qquad (4.1)$$

for the period of relaxation oscillation of the Liénard equation

$$\varepsilon x''+(ax^2-b)x'+x+sx^3=0, \qquad a\cdot b\cdot c\succ 0,\quad 0\prec\varepsilon\prec\prec 1 \qquad (4.2)$$

This equation (3.2) reduces to the Van der Pol equation

$$\varepsilon\frac{d^2y}{dt^2}-(1-y^2)\frac{dy}{dt}+y=0,\ \ 0\prec\varepsilon\prec\prec 1 \qquad (4.3)$$

for a = b = 1 and c = 0; and putting these in (4.1),we have that as $c\to 0$

$$\log\left(\frac{a+4bc}{a+bc}\right)=\log\left(1+\frac{3bc}{a+bc}\right)=\frac{3bc}{a+bc}-\frac{1}{2}\left(\frac{3bc}{a+bc}\right)^2+\frac{1}{3}\left(\frac{3bc}{a+bc}\right)^3=\frac{3bc}{a+bc},\ c\prec\prec 1 \text{ and so}$$

$$\begin{aligned}T(\varepsilon)&=\frac{a+bc}{c}\cdot\frac{3bc}{a+bc}-2b\log 2+\frac{3(a+3bc)}{a+4bc}\varepsilon^{\frac{2}{3}}t_0+0(\varepsilon\log\varepsilon)\\&=3-2\log 2+3\varepsilon^{\frac{2}{3}}t_0+0(\varepsilon\log\varepsilon)\end{aligned} \qquad (4.4)$$

for a = b = c = 1, which fits the results of Cole (1968) for van der Pol's equation (4.3) using the method of matched asymptotic expansion and of LaSalle, Bogoliubov and Mitropolsky using phase plane analysis.

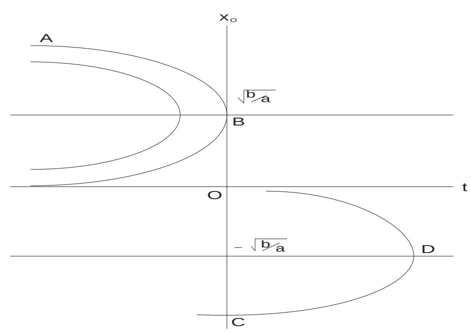

Fig. (1.1)

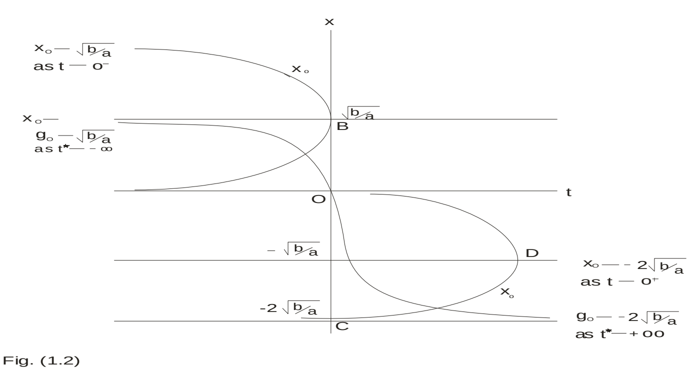

Fig. (1.2)

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