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Dynamic analysis of a Bernoulli-Euler beam via the Laplace transformation technique

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Abstract

In this paper the dynamic analysis of a simply supported Bernoulli-Euler beam subjected to a distributed load was investigated. The simplified form of the mathematical expression defining the dynamic displacement of the beam was formulated using the variational Indicator of the Hamiltonian principle. The method of Integral Transformation was used to obtain the series solution for the governing equation. The effect of the various beam parameters on dynamic deflection profile of the beam was simulated, it was observed that the contribution is mainly done by the first mode and higher modes of vibration can be neglected.

1.0 Introduction

One-dimensional continuous dynamics models lead to Partial Differential Equations (PDE) of motion. In particular, Partial Differential Equations arise when the generalized coordinate is a function of two (or more) variables. Beam elements may store kinetic energy by the transverse translation of its mass and (or) by the rotation of its rotary inertia about an axis perpendicular to the plane, and it may store strain energy by its bending deformation and (or) its shearing deformation. Several beam models that account for various combinations of this energy storage mechanism have been used by analyst to account for specific aspects of a dynamic phenomenon.

The problem of vibrations of elastic structures beam under the action of loads has been the subject of research by several mathematicians, physicist and engineers. Initially, the problem of elastic beam subject to loading, originated mainly from the applications in the field of transportation such as bridges, railways and buildings such as floors, etc

Ayre et al, [7] studied the effect of the ratio of the weight of the load to weight of a simply supported beam for a constantly moving mass load. They also obtained the exact solution for the infinite series. Kenney [3] found the possible velocities for the propagation of free bending waves and studied their relation to the critical velocity of the beam. He also presented an analytic solution and resonance diagrams for a constant velocity of a rapidly moving load on an elastic foundation including the effect of viscous damping. Lee et al, [8] developed a finite element model for static and free vibration analysis of a compressed beam resting on an elastic foundation. They obtained accurate solutions for uniform, ramp load with minimum number of element.

In most of those works that deals with distributed loads, the weight of the beam is not included as part of the load. Hence the purpose of our study is to focuses on the dynamic response of a Bernoulli-Euler beam with a distributed load subject to a tranverse excitation as with Aiyesimi, [1], where the weight of the beam is also considered as part of the load. The governing differential equation of motion is derived using the familiar Hamiltonian principle. In analyzing the resultant equation of motion according to Kreysiz [2] we invoke the finite Fourier transformation on the spatial coordinate and the Laplace transformation on the time coordinate to obtain the results.

2.0 Mathematical formulation

The following is a free-body diagram of a Bernoulli-Euler beam with a uniformly distributed load**:**

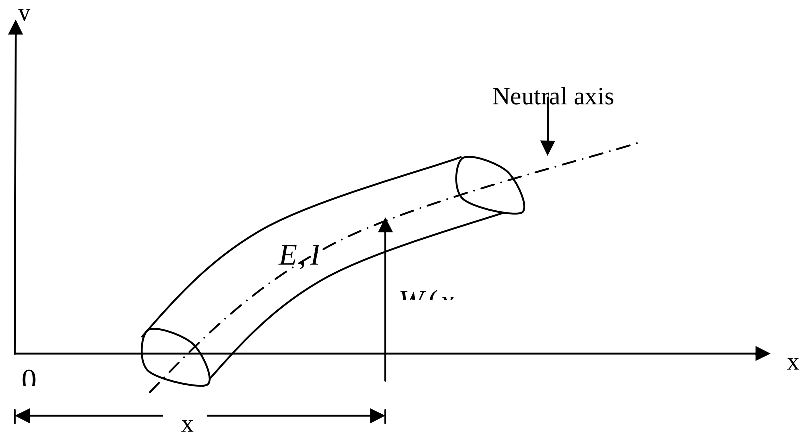

Figure 1 Displacement of a Bernoulli-Euler Beam

$W(x,t)$ - Generalized coordinate

According to Spiegel [4] for a beam undergoing flexural deformation we have $\gamma_{xy} = 0$. Also

$$V = \iiint_{vol} \left[\frac{E \varepsilon_x^2}{2} \right] dxdydz \qquad (2.1)$$

From technical mechanics of solids for Bernoulli-Euler beam $\varepsilon_x = \frac{-M_b Y}{EI}$.(2.2)

and

$$\frac{1}{\bar{p}} = \frac{\frac{\partial^2 w}{\partial x^2}}{\left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]^{3/2}} = \frac{M_b}{EI} \qquad (2.3)$$

M$_b$ – Bending moment

$\bar{\rho}$ - Radius of curvature of the beam in the plane. eqn. (2.3) can be rewritten as

$$\frac{M_b}{EI} \approx \frac{\partial^2 w(x,t)}{\partial x^2} \qquad (2.4)$$

given that $\left(\frac{\partial w}{\partial x}\right)$ is approximately unity and substituting equation. (2.4) into (2.2) we obtain

$$\in_x = -y \frac{\partial^2 w(x,t)}{\partial x^2} \qquad (2.5)$$

By virtue of equation (2.5), (2.1) becomes; $V = \iiint_{VOL} \frac{E}{2} \left[\frac{\partial^2 w(x,t)}{\partial x^2} \right]^2 y^2 dxdydz$ (2.6)

i.e. $$V = \int_0^l \frac{E}{2} \left[\frac{\partial^2 w(x,t)}{\partial x^2} \right]^2 \left[\int_y \int_z y^2 dydz \right] dx$$

According to Timoshenko [5], $\int_y \int_z y^2 dydz = I$

$$\therefore V = \int_o^l \frac{EI}{2} \left[\frac{\partial^2 w(x,t)}{\partial x^2} \right]^2 dx \qquad (2.7)$$

Equation (2.7) is the strain energy function for linearly elastic Bernoulli–Euler beam. Hence following Kenny [3], the governing differential equation of motion is therefore;

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+\rho A\frac{\partial^2 w}{\partial t^2}= f(x,t)\quad .0<x<l \tag{2.8}$$

and the corresponding natural boundary conditions given as;

$$M\frac{\partial^2 w}{\partial t^2}+Kw=\frac{\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+F(t)\ \ at\ \ x=l \text{ and } EI\frac{\partial^2 w}{\partial x^2}=0\ \ at\ \ x=l \tag{2.9}$$

where

- E – Young modulus
- I – Planner moment of inertia of the cross-section
- EI – Flexural Stiffness
- ρ - Density
- A – Cross sectional area
- $w(x,t)$ – Transversal displacement
- $f(x,t)$ –loading (distributed force)

We consider in this study the dynamic response of a simply-supported Bernoulli-Euler beam subjected to a distributed load $f(x,t)$ given as;

$$f(x,t)=\overline{P}\left(1-\frac{x}{l}\right)\sin\overline{\omega}t \tag{2.10}$$

where $\overline{P}$ -Uniform loading/unit-length; x – spatial coordinate; l – Length of beam, $\overline{\omega}$ - Circular frequency of harmonic forcing function; t – Time

Following Aiyesimi [1] a Bernoulli-Euler Beam with simple end conditions satisfy the boundary conditions: $\left[w(x,t),\frac{\partial^2 w}{\partial x^2}\right]_{x=0}=0$ and $\left[w(x,t),\frac{\partial^2 w}{\partial x^2}\right]_{x=l}=0$. The corresponding initial condition for the problem is given as; $\left[w(x,t);\frac{\partial^2 w(x,t)}{\partial t^2}\right]_{t=0}=0$

3.0 Solution Technique

Taking the Finite Fourier Sine transformation of the governing equation of motion with respect to the spatial variable x we have;

$$EI\frac{n^4\pi^4}{l^4}\overline{w}(x,t)+\rho A\frac{d^2\overline{w}}{dt^2}=\frac{\overline{P}\,l}{n\pi}\sin\overline{\omega}t \tag{3.1}$$

The Laplace transform of the above with respect to the time variable yields;

$$EI\frac{n^4\pi^4}{l^4}\tilde{\overline{w}}(n,t)+s^2\rho A\tilde{\overline{w}}(n,t)=\frac{\overline{P}\,l}{n\pi}\left[\frac{\overline{\omega}}{s^2+\overline{\omega}^2}\right] \tag{3.2}$$

To obtain the Laplace inverse of equation (3.2) we have

$$L^{-1}\{\overline{f}(s)\}=\frac{\overline{P}\,l}{Mn\pi}Sin\overline{\omega}t,\quad L^{-1}\{\overline{g}(s)\}=\frac{Sin\alpha t}{\alpha}$$

The Laplace inverse transform of equation (3.2) is the Convolution denoted by the integral.

$$\int_0^t f(t-u)g(u)du \tag{3.3}$$

$$\overline{w}(n,t)=\frac{\overline{P}\,l}{\alpha Mn\pi}\int_0^t Sin\overline{\omega}(t-u)Sin\alpha u du \tag{3.4}$$

$$\overline{w}(n,t) = \frac{\overline{P}\, l}{2\alpha\, Mn\,\pi}\left[\frac{\cos\varpi t}{(\alpha+\varpi)}\sin(\alpha+\varpi)t - \frac{\sin\varpi t}{(\alpha+\varpi)}\cos(\alpha+\varpi)t\right.$$
$$\left. + \frac{\sin\varpi t}{(\alpha+\varpi)} - \frac{\cos\varpi t}{(\alpha-\varpi)}\sin(\alpha-\varpi)t - \frac{\sin\varpi t}{(\alpha-\varpi)}\cos(\alpha-\varpi)t + \frac{\sin\varpi t}{(\alpha-\varpi)}\right]$$

Finally, on taking the inverse Fourier transform of $\overline{w}$ we have;

$$w(x,t) = \frac{2\overline{P}\, l^4}{EI\,\pi^5}\sum_{n=1}^{\infty}\frac{K_1}{n^5}\sin\frac{n\pi}{l}x\left[(\sin\varpi t - \frac{\varpi}{\alpha}\sin\varpi t\right] \qquad (3.5)$$

where

$$\alpha_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{EI}{\rho A}}.,\ \mathrm{K}_1 = \left(1 - \frac{\varpi^2}{\alpha^2}\right)^{-1}$$

4.0 Numerical Simulation

The displacement profiles of the beam are displaced graphically in what follows demonstrating the effect flexural rigidity and the damping parameter on the amplitude of vibration.

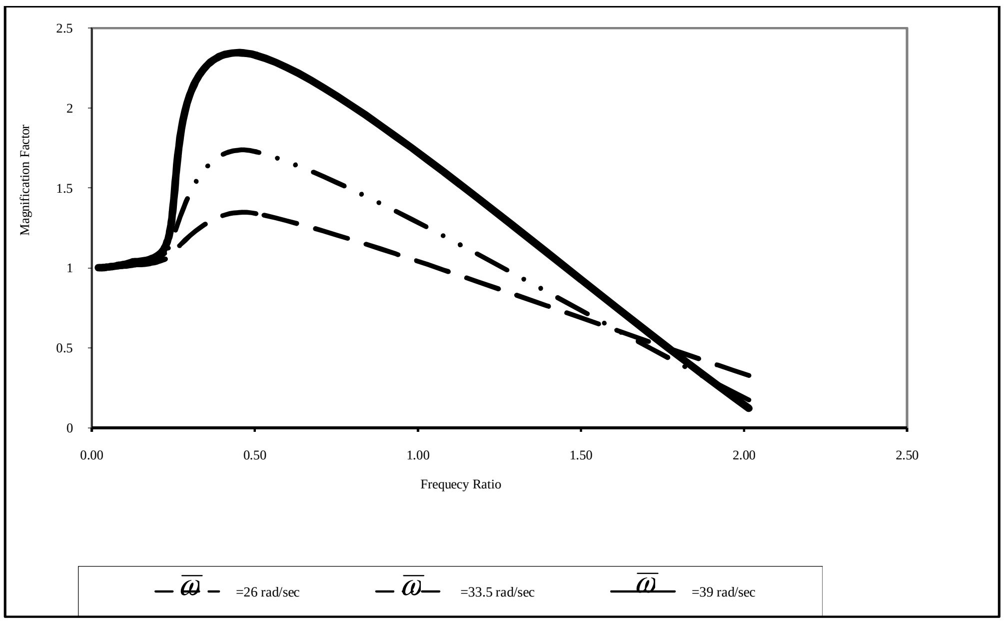

Figure 2: Variation of dynamic Magnification factor with three different frequencies

5.0 Discussion of results and conclusion

From the dynamic profile of the beam it is observed that the beam has an infinite number of degree of freedom and more than one mode of vibration may exist with each mode having a different natural frequency. The lowest natural frequency of the beam corresponds to the fundamental of the frequency of the beam, while the corresponding mode of vibration is found to be the fundamental mode of vibration which conforms with the results of Kenny [3]

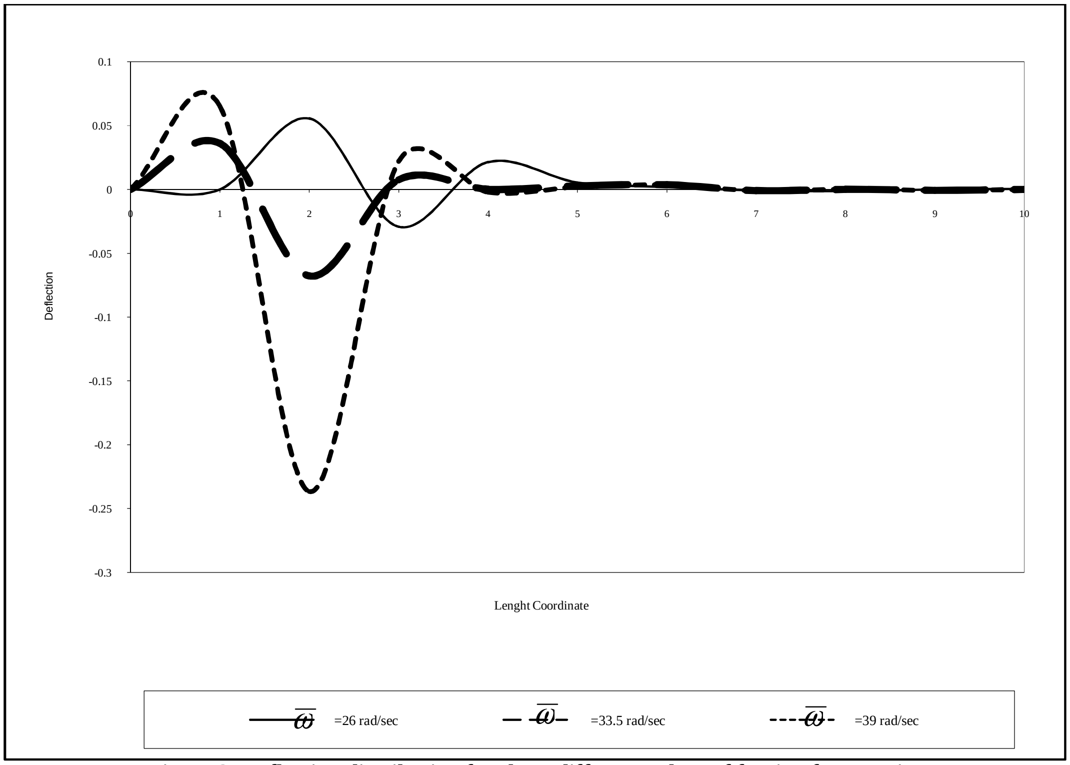

Figure 3: Deflection distribution for three different values of forcing frequencies

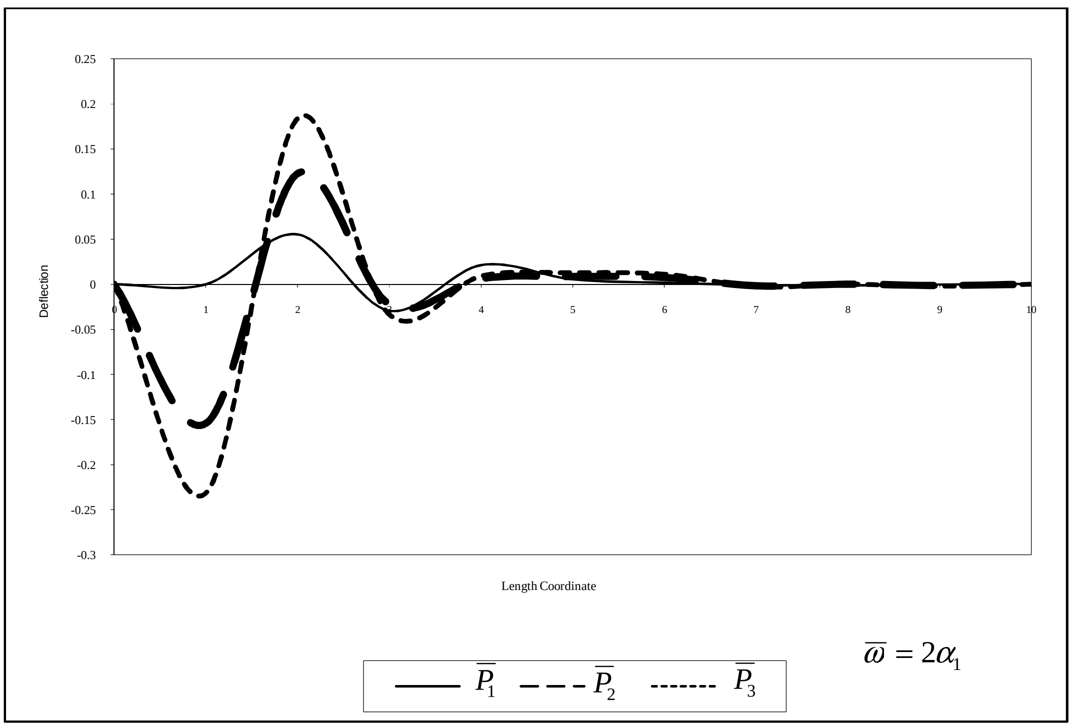

Figure 4: Deflection distribution for different values of uniform loading/unit length

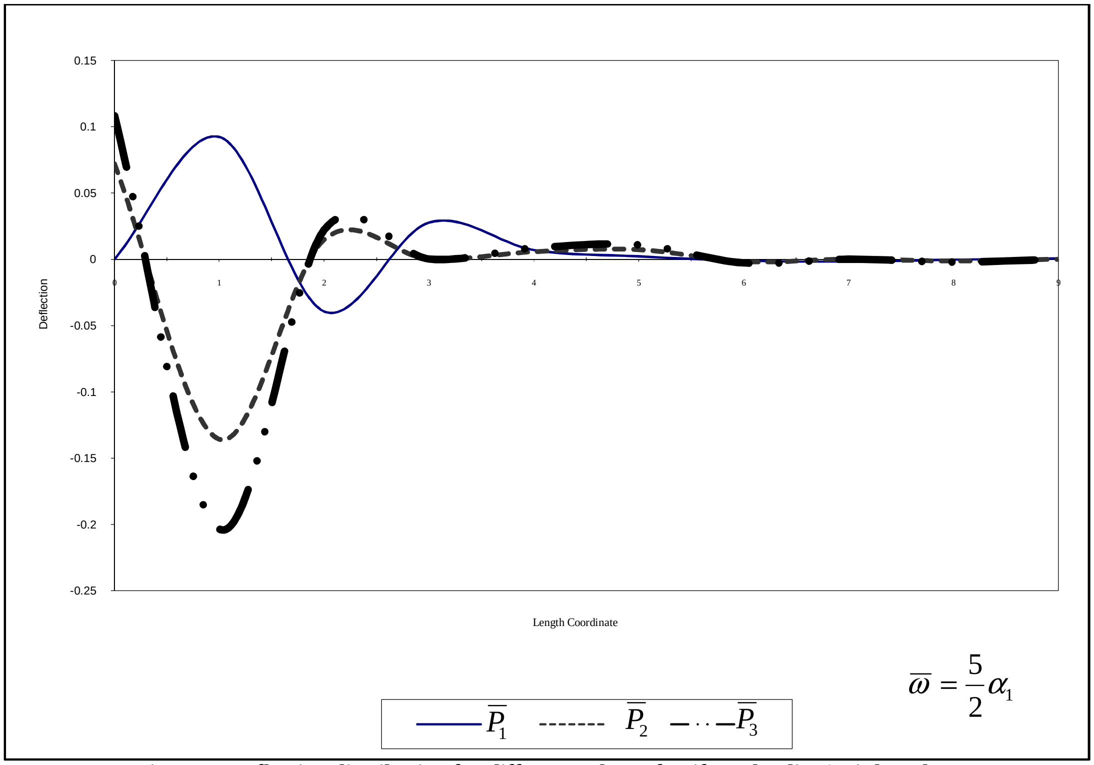

Figure 5: Deflection distribution for different values of uniform loading/unit length

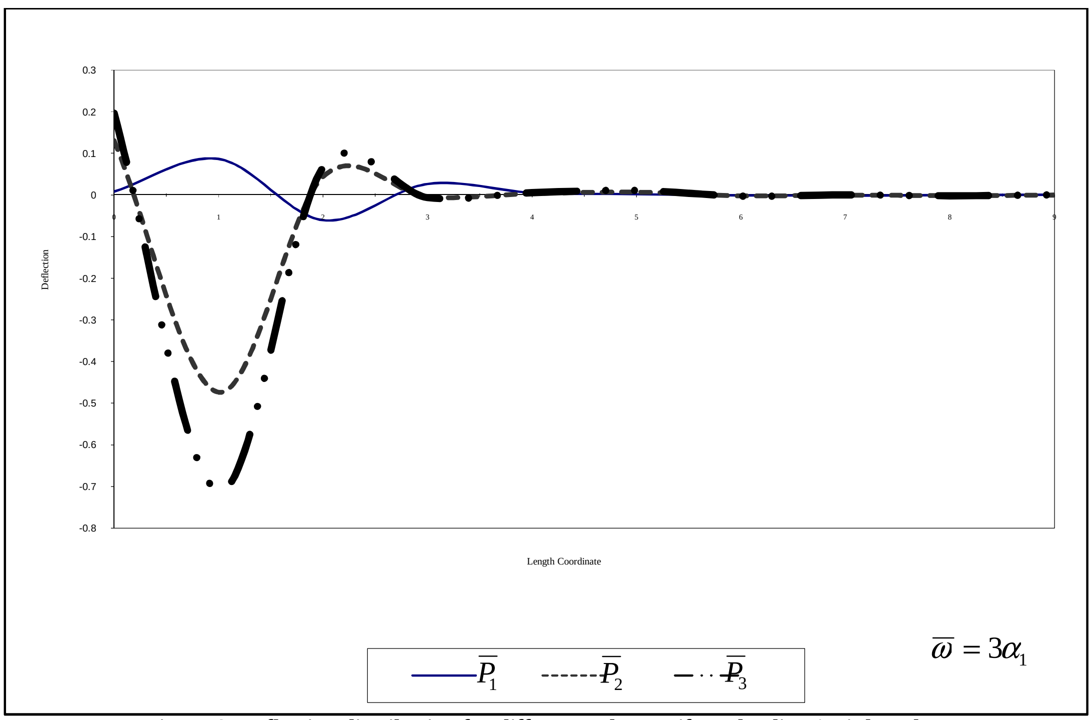

Figure 6: Deflection distribution for different values uniform loading /unit length

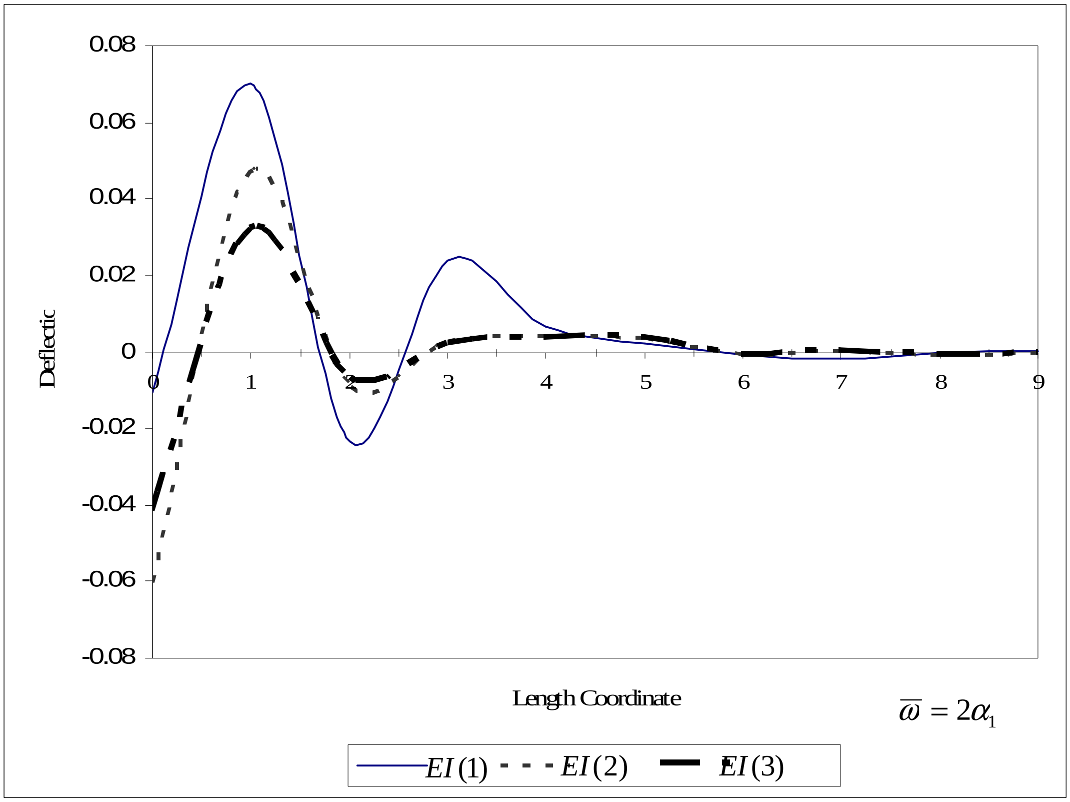

Figure: .7 Deflection distribution for different values of stiffness

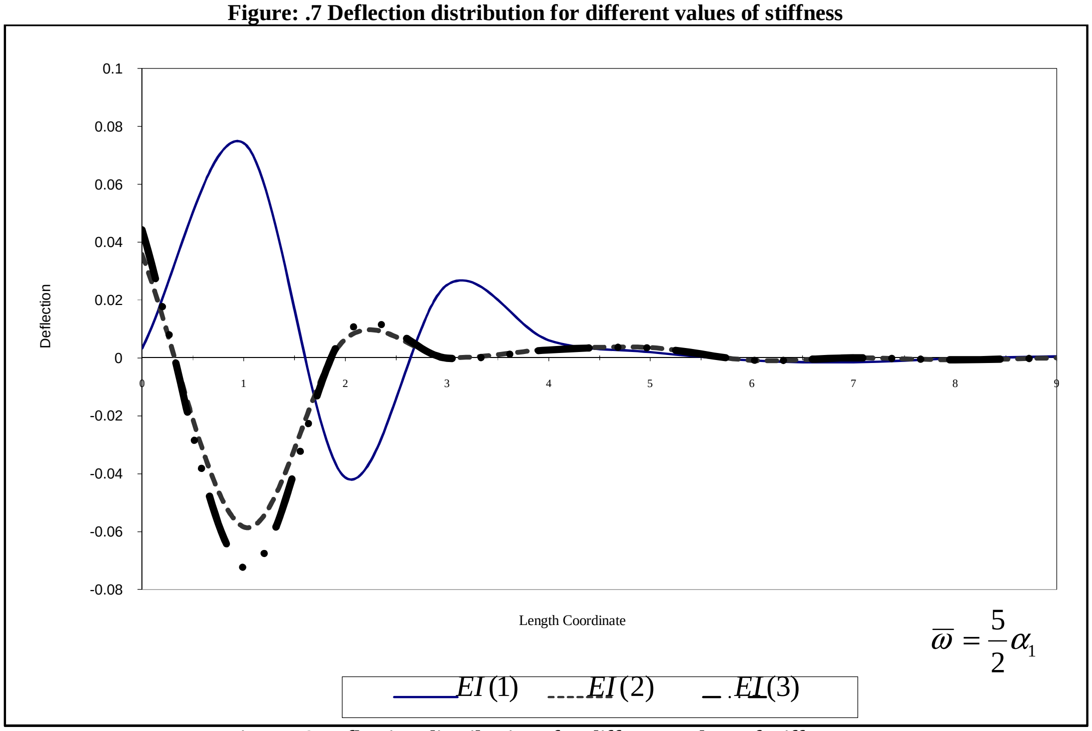

Figure: 8 Deflection distributions for different values of stiffness

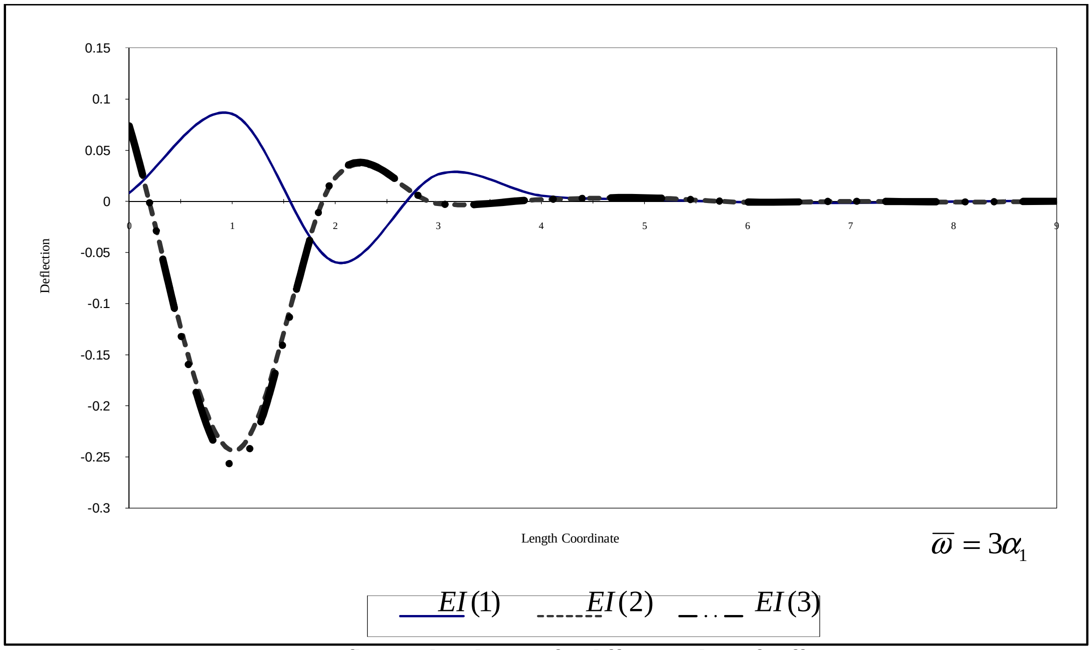

Figure 9: Deflection distributions for different values of stiffness

The resonance condition occurs for the beam when the forcing frequency of the load corresponds to the natural frequency of the beam as obtained by Spiegel [4]. The results also show that as the value of the forcing frequency $(\overline{\omega})$ approaches that of natural frequency (α) i.e. as $\frac{\overline{\omega}}{\alpha}$ approaches unity the magnification factor grows rapidly, and it's value at or near resonance is very sensitive to changes in the amount of damping. The maximum value of the magnification factor occurs for a value of $\frac{\overline{\omega}}{\alpha}$ slightly less than unity. It is also established as found in Kreyszig [2] that the higher modes of vibration are generally neglected when considering a distributed load. The result of this work shows that, this is because the contribution to dynamic displacement is mainly done by the first mode and also the characteristic shape of the first mode is similar to the load distribution.

Also it was observed that the actual beam shape can be assumed by truncation in the series solution obtained, and the harmonic of the forcing frequency are multiple of the forcing frequency and they can cause resonance if they correspond to the natural frequency of the system. Finally it was observed that the amplitude of vibration increases with increasing forcing frequency.

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