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# On the dynamic Stability of a quadratic-cubic elastic model structure pressurized by a slowly varying load

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**Abstract**

**The main substance of this investigation is the determination of the dynamic buckling load of an imperfect quadratic-cubic elastic model structure , which ,in itself, is a Mathematical generalization of some of the many physical structures normally encountered in engineering practice and allied fields. The load function in which the time variable is explicitly expressed, varies very slowly over a natural period of oscillation of the structure. The nonlinearity is quadratic-cubic in nature and multiple-scaling two-timing regular perturbation technique is utilized. The result shows that the dynamic buckling load depends on the first derivative of the load function evaluated at the initial time .Besides , it is established that it is possible to relate the dynamic buckling load to its static equivalent and this by-passes the labour of repeating the entire arduous process for different imperfection parameters .**

## 1.0 Introduction

The degree of dynamic stability of any elastic structure stressed by any dynamic load is an important stability criterion that determines the suitability or otherwise of any such structure for practical purposes. Over the years, many and diverse time dependent loading histories into which these structures have been subjected , have been prescribed and actually used to analyse the state of dynamic stability of these structures . A cautious observation of these loading histories indicates that they fall into two main categories, namely, those in which the time variable is implicitly expressed and those in which the time variable is explicitly expressed . Loading histories of the first category include step loading, impulsive loading and rectangular loading among others. Examples of the second category include periodic and triangular loadings .Others in this category include specific examples such as the ones considered by Svalbonas and Kalnins [1] and Aksogan and Sofiyev [2] among others .The novelty in the first category is the relative abject simplicity of the resultant autonomous system. This , infact necessitated an easy determination by Budiansky and Hutchinson [3,4] ,of the dynamic buckling loads of some simple elastic structures , using phase plane analysis .On the contrary, the second category of loading history always leads to non-autonomous system, the solutions which are always relatively more involving .Our problem at hand is an example of the second category . Here, the time variable is explicitly expressed and the loading history is dynamically slowly varying over a natural period of oscillation of the imperfect elastic structure.

## 2.0 Formulation

A simple structural device that amply captures the essence of our objective, is a two-arm simply-supported quadratic-cubic imperfect column stressed by, for now, a time dependent load $F(T)$, applied just after the initial time T=0 .The column is assumed rigid and weightless and carries a mass $M$ at the central hinge (see figure) .Each of the two arms of the column is of length $L$ . The motion of $M$ is restrained by the action of a quadratic-cubic spring (ie having a quadratic-cubic nonlinearity) that provides a restoring force of $KL\left(X - \alpha X^2 - \beta X^3\right)$ where $\alpha$ and $\beta$ are constants and are such that

$\alpha > 0$, $\beta > 0$ and $K$ is the spring constant considered positive .Here $X$ is the central hinge displacement from the equilibrium position. Assuming small angular displacements characterized by $\cos\phi \cong 1$ , $\sin\phi \cong \phi$ ,the relevant equations of dynamic equilibrium are easily seen to be

$$M \frac{d^2 X}{dT^2} + KL\left(1 - \frac{2F(T)}{KL^2}\right)X - KL\left(\alpha X^2 + \beta X^3\right) = \frac{2\bar{X}F(T)}{L}, \; T > 0 \qquad (2.1a)$$

$$X(0) = \frac{dX(0)}{dT} = 0 \qquad (2.1b)$$

where $\bar{X}$ is the initial displacement which shall serve as initial imperfection . In the above formulation, we have neglected all nonlinear geometric terms of the same order as $X^3$ (and higher) compared to that of the spring characteristics. We now assume the following nondimensional terms: $\xi = \frac{X}{L}$, $\tilde{t} = T\sqrt{\frac{KL}{M}}$ , $\lambda = \frac{2F(0)}{KL^2}$, $f(\delta\tilde{t}) = \frac{F(T)}{F(0)}$, $\in = \frac{\bar{X}}{L}$ . Thus, the nondimensional forms of (2.1a,b) are

$$\frac{d^2\xi}{d\tilde{t}^2} + \left(1 - \lambda f(\delta\tilde{t})\right)\xi - a\xi^2 - b\xi^3 = \lambda \in f(\delta\tilde{t}), \; \tilde{t} > 0 \qquad (2.2a)$$

$$\xi(0) = \frac{d\xi(0)}{d\tilde{t}} = 0 \qquad (2.2b)$$

Here, we have $a = L\alpha$, $b = L^2\beta$, $a > 0$, $b > 0$ and $\lambda$ is a nondimensional load parameter satisfying the inequality $0 < \lambda < 1$ while $\delta$ and $\in$ are two small parameters satisfying the inequalities $0 < \delta << 1$ and $|\in| << 1$. At the same time, the term $f(\delta\tilde{t})$ is a slowly varying continuous load function of time $\tilde{t}$ , having right hand derivatives of all orders at $\tilde{t} = 0$ and satisfying the conditions

$$f(0) = 0 \;, \left|f(\delta\tilde{t})\right| \le 1 \;\; for \;\; \tilde{t} > 0 \qquad (2.3)$$

The function $f(\delta\tilde{t})$ varies very slowly over a natural period of vibration of the structure. Except for condition (2.3), $f(\delta\tilde{t})$ is strictly arbitrary .The formulation enunciated here was formally begun in [5]. In [5], the small parameters $\delta$ and $\in$ were assumed to be mathematically unrelated. Thus, under such an assumption, (2.2a) is a two-small-parameter strongly nonlinear differential equation with explicitly expressed time dependent and dynamically slowly varying coefficients. The objective in [5] (and, indeed, in the present investigation) was the determination of a particular value of $\lambda$ namely $\lambda_D$, called the dynamic buckling load , in which the imperfect elastic structure buckles dynamically . We however remark that there are several physical structures of utmost engineering and structural importance in which (2.2a) is a generalization. These include (a) imperfect columns ( finite and infinite) on elastic nonlinear quadratic-cubic foundations, (b) imperfect cylindrical shells and (c) imperfect toroidal shells , among others . The last two examples hold if the buckling mode is assumed not only in the shape of the imperfection but also in some higher multiples of the imperfection.

The present investigation is the first of a three-part study of the same dynamical system represented by (2.2a,b) under various restrictions on the parameters . Where as in [5], $\delta$ and $\in$ were deemed mathematically non-related, in the present (and two subsequent investigations that will immediately follow), this Mathematical unrelatedness is relaxed. Various Mathematical relationships between $\delta$ and $\in$ will be assumed and the resultant dynamic buckling loads in each relationship will be determined. The dynamic buckling load depends partly on (i) the unrelatedness of $\delta$ and $\in$ (if indeed they are unrelated ) and partly on ,(ii) the functional relationship between the two parameters when they are related .The Mathematical sophistication in each case is slightly different .In this investigation ,we shall assume the linear relationship

$$\delta = \in \qquad (2.4)$$

The introduction of (2.4) into (2.2a,b) yields

$$\frac{d^2\xi}{d\tilde{t}^2}+\left(1-\lambda f(\in\tilde{t})\right)\xi - a\xi^2 - b\xi^3 = \lambda\in f(\in\tilde{t}),\ \tilde{t}>0 \tag{2.5a}$$

$$\xi(0) = \frac{d\xi(0)}{d\tilde{t}} = 0 \tag{2.5b}$$

Unlike (2.2a,b), (2.5a,b) give a one-parameter nonlinear oscillatory system with explicitly time dependent slowly varying coefficients . We shall assume

$$f(0)=1\ ,\left|f(\in\tilde{t})\right|\le 1 \quad \text{for} \quad \tilde{t}>0 \tag{2.6}$$

## 3.0 Static buckling load $\lambda_s$

To analyse this case, we ignore the inertia term in (5a) and set $f(\in\tilde{t})\equiv 1$. Thus , we get

$$(1-\lambda)\xi - a\xi^2 - b\xi^3 = \lambda\in \tag{3.1}$$

As in [3,4] the static buckling load $\lambda_s$ is obtained from the maximization $\frac{d\lambda}{d\xi}=0$. This gives

$$(1-\lambda_s)-2a\xi_s-3b\xi_s^2=0 \tag{3.2}$$

where $\xi_s$ is the value of $\xi$ at buckling. Thus we get

$$\xi_s^2=\frac{(1-\lambda_s)-2a\xi_s}{3b},\ \xi_s=\frac{-a\pm\sqrt{3b}(1-\lambda_s)^{\frac{1}{2}}R^{\frac{1}{2}}}{3b}\ ,\ \ R=\left(\frac{a^2}{3b(1-\lambda_s)}\right), \tag{3.3}$$

We shall however take the negative sign of the two signs from square root .The positive sign has no physical relevance in this circumstance .On evaluating (3.1) at buckling we obtain

$$\left[\left\{\left(\frac{(1-\lambda_s)}{3}\right)+\frac{a^2}{9b}\right\}\left\{\frac{\sqrt{3b}\,(1-\lambda_s)^{\frac{1}{2}}R^{\frac{1}{2}}}{3b}-\frac{a}{3b}\right\}\right]=\frac{\lambda_s\in}{2}+\frac{a(1-\lambda_s)}{18} \tag{3.4}$$

To arrive at (3.4), we have substituted for $\xi_s^2$ from (3.3) into (3.1) in two quick successions and lastly, substituted for $\xi_s$ from the second of (3.1). The result (3.4) is implicit in the load parameter $\lambda_s$ .When $\in=0$ in (3.1), we have an eigen value problem with solutions $\xi=0$ for all $\lambda$ and $\lambda=1-a\xi-b\xi^2$. The case $\lambda=1$ Is called the classical buckling load and is usually represented by $\lambda_c$. Thus we have $\lambda_c=1$

## 4.0 Step loading case.

The dynamic buckling load $\lambda_D$ for the step loading case is obtained by setting $f(\in\tilde{t})=1$ in (5a) and getting the following by so doing

$$\frac{d^2\xi}{d\tilde{t}^2}+(1-\lambda)\xi-a\xi^2-b\xi^3=\lambda\in \tag{4.1a}$$

$$\xi(0)=\frac{d\xi(0)}{d\tilde{t}}=0 \tag{4.1b}$$

The system (4.1a) is an autonomous case and a direct integration gives

$$\frac{1}{2}\left(\frac{d\xi}{d\tilde{t}}\right)^2+\left(\frac{1-\lambda}{2}\right)\xi^2-\frac{a\xi^3}{3}-\frac{b\xi^4}{4}=\xi\lambda\in \tag{4.2}$$

A schematic appraisal of (4.2) shows that at the origin and its immediate neighbourhood , the trajectories are elliptical, resulting in bounded solutions. The elliptical trajectories increase in size from the origin with increased load $\lambda$ and time $\tilde{t}$ . At time $\tilde{t}_D$ and load $\lambda_D$ ( the dynamic buckling load), the largest elliptical trajectory with periodic and hence bounded solution occurs subsequent upon which every other trajectory is hyperbolic with monotonically increasing and hence unbounded solutions .

We define $\lambda_D$ as the largest load parameter for which bounded solutions occur . At maximum displacement $\xi_m$, we get

$$\frac{(1-\lambda)\xi_m}{2}-\frac{a\xi_m^2}{3}-\frac{b\xi_m^3}{4}=\lambda\in \tag{4.3a}$$

Simplifying (4.3a) further, we get

$$\lambda=\frac{6\xi_m-4a\xi_m^2-3b\xi_m^3}{6(2\in+\xi_m)} \tag{4.3b}$$

As in [1, 2], the condition for dynamic buckling is

$$\frac{d\lambda}{d\xi_m}=0 \tag{4.4}$$

On performing (4.4) ,using (4.3a)or (4.3b),we get

$$\frac{(1-\lambda_D)}{2}-\frac{2a\xi_{mD}}{3}-\frac{3b\xi_{mD}^2}{4}=0 \tag{4.5}$$

where $\xi_{mD}$ is the value of $\xi_m$ at buckling ,that is, $\xi_{mD}=\xi_m(\lambda_D)$. The solution of (4.5) is

$$\xi_{mD}^2=\frac{6(1-\lambda_D)-8a\xi_{mD}}{9b}\ ,\ \xi_{mD}=\frac{-4a\pm3\sqrt{6b}\,(1-\lambda_D)^{\frac{1}{2}}Q^{\frac{1}{2}}}{9b}\quad; \tag{4.6}$$

$$Q=1+\frac{8a^2}{27b(1-\lambda_D)}$$

If we evaluate (4.3a) at buckling by substituting, first, for $\xi_{mD}^2$ from the first of (4.6) in two quick successions and lastly for $\xi_{mD}$ from the second of (4.6), we get

$$\left[\left\{\frac{(1-\lambda_D)}{3}+\frac{32a^2}{81b}\right\}\left\{\frac{\sqrt{6b}\ Q^{\frac{1}{2}}(1-\lambda_D)^{\frac{1}{2}}}{3b}-\frac{4a}{9b}\right\}\right]=\lambda_D\in+\frac{8a(1-\lambda_D)}{27b} \tag{4.7}$$

As in (3.4), we observe that (4.7) is implicit in the load parameter $\lambda_D$ . We can eliminate the small parameter $\in$ in (4.7) using (3.4) to get

$$\left(\frac{\lambda_D}{\lambda_s}\right)=\frac{\left[\left\{\frac{(1-\lambda_D)}{3}+\frac{32a^2}{81b}\right\}\left\{\frac{\sqrt{6b}\ (1-\lambda_D)^{\frac{1}{2}}Q^{\frac{1}{2}}}{3b}-\frac{4a}{9b}\right\}-\frac{8a(1-\lambda_D)}{27b}\right]}{2\left[\left\{\frac{(1-\lambda_s)}{3}+\frac{a^2}{9b}\right\}\left\{\frac{\sqrt{3b}\ (1-\lambda_s)^{\frac{1}{2}}R^{\frac{1}{2}}}{3b}-\frac{a}{3b}\right\}-\frac{a(1-\lambda_s)}{18b}\right]} \tag{4.8}$$

Thus we can determine the dynamic buckling load $\lambda_D$ of the structure from a knowledge of the static buckling load $\lambda_s$ .This by-passes the labour of repeating the entire process for different imperfection amplitudes $\in$ .

## 5.0 Slowly varying load

For clarity of further analysis, we recast (2.5a,b) as

$$\frac{d^2\xi}{d\tilde{t}^2}+(1-\lambda f(\in\tilde{t}))\xi-a\xi^2-b\xi^3=\lambda\in f(\in\tilde{t}),\ \tilde{t}>0 \tag{5.1}$$

$$\xi(0)=\frac{d\xi(0)}{d\tilde{t}}=0 \tag{5.2}$$

The problem (5.1, 5.2), wearing a semblance of seemingly relative simplicity compared to (2.2ab), is here solved using multiple scaling-two–timing regular perturbation technique .Similar works were done by Wang and Tian [6-8] . We now let

$$\tau = \in \tilde{t}, \quad \frac{dt}{d\tilde{t}} = \left(1 - \lambda f(\in \tilde{t})\right)^{\frac{1}{2}} \tag{5.3}$$

Based on (5.3), the displacement $\xi(\tilde{t})$ can now be thought of as a function of the fast time *t* and slow time $\tau$ .Thus we have

$$\frac{d\xi}{d\tilde{t}} = (1-\lambda f)^{\frac{1}{2}}\xi_t + \in \xi_\tau \tag{5.4a}$$

$$\frac{d^2\xi}{d\tilde{t}^2} = (1-\lambda f)\,\xi_{tt} + 2\in(1-\lambda f)^{\frac{1}{2}}\xi_{t\tau} + \in^2 \xi_{\tau\tau} - \frac{\lambda f'\xi_t}{2(1-\lambda f)^{\frac{1}{2}}} \tag{5.4b}$$

where $(\ )' = \frac{d(\ )}{dt}$ and $(\ )_t$ and $(\ )_\tau$ indicate partial differentiation with respect to t and $\tau$ respectively. We now let

$$\xi(\tilde{t}) = \xi(t,\tau) = \sum_{i=1}^{\infty}\zeta^{(i)}(t,\tau)\in^i \tag{5.5}$$

and substitute (5.4a,b) into (5.1, 5.2), using (5.5), and after ,equate the coefficients of integral powers of $\in$ and get

$$L\zeta^{(1)} \equiv \zeta^{(1)}_{tt} + \zeta^{(1)} = B(\tau)\,,\ B(\tau) = \frac{\lambda f}{1-\lambda f} \tag{5.6}$$

$$L\zeta^{(2)} = -2(1-\lambda f)^{-\frac{1}{2}}\zeta^{(1)}_{t\tau} + \frac{\lambda f'(1-\lambda f)^{-\frac{3}{2}}\zeta^{(1)}_t}{2} + a\left(\zeta^{(1)}\right)^2(1-\lambda f)^{-1} \tag{5.7}$$

$$L\zeta^{(3)} = -2(1-\lambda f)^{-\frac{1}{2}}\zeta^{(2)}_{t\tau} + \frac{\lambda f'(1-\lambda f)^{-\frac{3}{2}}\zeta^{(2)}_t}{2} + 2\zeta^{(1)}\zeta^{(2)}(1-\lambda f)^{-1} + b\left(\zeta^{(1)}\right)^3(1-\lambda f)^{-1} - \zeta^{e(1)}_{\tau\tau}(1-\lambda f)^{-1} \tag{5.8}$$

The initial conditions which are evaluated at $(t,\tau) = (0,0)$ are given by

$$\zeta^{(i)} = 0\,,\ \zeta^{(1)}_t = 0\,,\ i = 1,2,3,\cdots \tag{5.9a}$$

$$\zeta^{(k)}_t + (1-\lambda)^{-\frac{1}{2}}\zeta^{(i)}_\tau = 0\,,\ i = k-1,\ k = 2,3,4,\cdots \tag{5.9b}$$

The solution of (5.6) subject to (5.9a) for $i$ =1, is

$$\zeta^{(1)}(t,\tau) = \alpha_1(\tau)\cos t + \beta_1(\tau)\sin t + B \tag{5.10a}$$

$$\alpha_1(0) = -B(0) = -\frac{\lambda}{1-\lambda},\ \beta_1(0) = 0 \tag{5.10b}$$

We shall however use $B(0) = B_0$ .Thus we have

$$\alpha_1(0) = -B_0 \tag{5.10c}$$

We now substitute for $\zeta^{(1)}$ from (5.10a) into (5.7) and, to ensure a uniformly valid solution in the time scale *t*, equate to zero the resultant coefficients of cost and sint and obtain the following respective equations

$$\beta_1' - \frac{\lambda f'\beta_1}{4(1-\lambda f)} = \frac{\alpha_1 B}{(1-\lambda f)^{\frac{3}{2}}};\ \beta_1'(0) = -\frac{B_0^2}{(1-\lambda f)^{\frac{3}{2}}} \tag{5.11a}$$

$$\alpha_1' - \frac{\lambda f'\alpha_1}{4(1-\lambda f)} = -\frac{\beta_1 B}{(1-\lambda f)^{\frac{3}{2}}}\ ;\ \alpha'(0) = -\frac{B_0^2 f'(0)}{4} \tag{5.11b}$$

By multiplying the first of (5.11a) by $\beta_1$ and the first (5.11b) by $\alpha_1$, adding and solving the resultant equation in the dependent variable $\alpha_1^{(2)} + \beta_1^{(2)}$ , we get

$$\alpha_1^{(2)} + \beta_1^{(2)} = B_0^2\left(\frac{1-\lambda}{1-\lambda f}\right)^{\frac{1}{2}} \qquad (5.12)$$

where we have used (5.10b) . However, the full solution of (5.11a,b) (see appendix) yields

$$\alpha_1(\tau) = B_0\, e^{(M_1(\tau)-M_1(0))}\left[-\cos M_2 + \frac{1}{\Omega_0}\left\{\theta(0) - \frac{f'(0)\,B_0^2}{4}\right\}\sin M_2\right] \qquad (5.13a)$$

$$\beta_1(\tau) = -B_0\, e^{(M_1(\tau)-M_1(0))}\sin M_2 \qquad (5.13b)$$

where

$$M_1(\tau) = -\frac{1}{4}\ln(1-\lambda f),\ M_2(\tau) = \int_0^\tau \left\{\frac{B(s)}{(1-\lambda f(s))^{\frac{3}{2}}}\right\} ds\ ;\ \Omega(\tau) = M_2'(\tau) = \frac{B(\tau)}{(1-\lambda f)^{\frac{3}{2}}}, \qquad (5.13c)$$

$$\theta(\tau) = \frac{\lambda f'}{4(1-\lambda f)}\ ,\quad \theta(0) = \frac{B_0 f'(0)}{4}\ ;\quad \Omega_0 = \Omega(0) = \frac{B_0}{(1-\lambda)^{\frac{3}{2}}} \qquad (5.13d)$$

The remaining equations in the substitution into (5.7) are

$$L\zeta^{(2)} = a\left(S_1 + S_2\cos 2t + S_3 \sin 2t\right) \qquad (5.14a)$$

$$\zeta^{(2)}(0,0) = 0,\quad \zeta_t^{(2)}(0,0) + (1-\lambda)^{-\frac{1}{2}}\zeta_\tau^{(1)}(0,0) = 0 \qquad (5.14b)$$

where

$$S_1(\tau) = \frac{\left(\alpha_1^2 + \beta_1^2 + 2B^2\right)}{2(1-\lambda f)},\ S_1(0) = \frac{3\,B_0^2}{2(1-\lambda)}\ ,\ S_1'(0) = \frac{B_0^3 f'(0)(7\lambda+8)}{4\lambda(1-\lambda)} \qquad (5.15a)$$

$$S_2(\tau) = \frac{\left(\alpha_1^2 - \beta_1^2\right)}{2(1-\lambda f)},\ S_2(0) = \frac{B_0^2}{2(1-\lambda)},\ S_2'(0) = \frac{3\,B_0^3 f'(0)}{4(1-\lambda)} \qquad (5.15b)$$

$$S_3(\tau) = \frac{(\alpha_1\beta_1)}{(1-\lambda f)}\ ,\quad S_3(0) = 0\quad ,S_3'(0) = \frac{B_0^3}{(1-\lambda)^{\frac{5}{2}}} \qquad (5.15c)$$

The solution of (5.14a,b), using (5.15a-c) is

$$\zeta^{(2)}(t,\tau) = \alpha_2(\tau)\cos t + \beta_2(\tau)\sin t + a\left[S_1 - \left(\frac{S_2\cos 2t + S_3 \sin 2t}{3}\right)\right] \qquad (5.16a)$$

$$\alpha_2(0) = -\frac{4aB_0^2}{3(1-\lambda)},\ \beta_2(0) = -\frac{B_0 f'(0)(4-\lambda)}{4(1-\lambda)^{\frac{3}{2}}} \qquad (5.16b)$$

In the next round of analysis, we shall need the following simplifications

$$\frac{2\zeta^{(1)}\zeta^{(2)}}{1-\lambda f} = \frac{2}{(1-\lambda f)}\left[S_4 + (S_5 + B\alpha_2)\cos t + (S_6 + B\beta_2)\sin t\right.$$
$$\left. + S_7\cos 2t + S_8 \sin 2t + S_9\cos 3t + S_{10}\sin 3t\right] \qquad (5.17a)$$

where

$$S_4(\tau) = \frac{\alpha_1\alpha_2}{2} + \frac{\beta_1\beta_2}{2} + aBS_1,\ S_4(0) = \frac{13aB_0^3}{6(1-\lambda)}\ ;$$

$$S_5(\tau) = a\left(\alpha_1 S_1 - \frac{\alpha_1 S_3}{6}\right) - \frac{a\,\beta_1 S_3}{6} \qquad (5.17b)$$

$$S_5(0) = -\frac{aB_0^3}{1-\lambda}\ ;\ S_6(\tau) = a\left(S_1\beta_1 - \frac{\alpha_1 S_3}{6} + \frac{\beta_1 S_2}{6}\right),\ S_6(0) = 0 \qquad (5.17c)$$

$$S_7(\tau) = a\left(\frac{\alpha_1\alpha_2}{2} - \frac{\beta_1\beta_2}{2} - \frac{aBS_2}{3}\right),\ S_7(0) = \frac{aB_0^3}{2(1-\lambda)};$$
$$S_8(\tau) = \left(\frac{\alpha_1\beta_2}{2} + \frac{\alpha_2\beta_1}{2} - aBS_3\right) \tag{5.17d}$$

$$S_8(0) = \frac{B_0^2 f'(0)(4-\lambda)}{8(1-\lambda)^{\frac{3}{2}}}\ ;\ S_9(\tau) = \frac{a}{6}(\beta_1 S_3 - \alpha_1 S_2),\ \ S_9(0) = S_7(0) \tag{5.17e}$$

$$S_{10}(\tau) = -\frac{a}{6}(\alpha_1 S_3 + \beta_1 S_2)\ ,\quad S_{10}(0) = 0 \tag{5.17f}$$

Similarly, we shall need

$$\frac{\left(\zeta^{(1)}\right)^3}{1-\lambda f} = \frac{(\alpha_1 \cos t + \beta_1 si/'nt + B)^3}{1-\lambda f} = \frac{1}{(1-\lambda f)}[S_{11} + S_{12}\cos t + S_{13}\sin t$$
$$+S_{14}\cos 2t + S_{15}\sin 2t + S_{16}\cos 3t + S_{17}\sin 3t] \tag{5.18a}$$

where

$$S_{11}(\tau) = \frac{3\alpha_1^2 B}{4} + B^3 + \frac{3B\beta_1^2}{2},\ \ S_{11}(0) = \frac{5B_0^3}{2};$$
$$S_{12}(\tau) = \frac{3\alpha_1^3}{4} + 3\left(B^2\alpha_1 + \frac{\alpha_1\beta_1^2}{4}\right) \tag{5.18b}$$

$$S_{12}(0) = -\frac{15B_0^3}{4}\ ;\ \ S_{13}(\tau) = \frac{3\alpha_1^2\beta_1}{4} + 3B\beta_1^2 + \frac{3\beta_1^3}{4}\ , S_{13}(0) = 0 \quad (3.18c)$$

$$S_{14}(\tau) = \frac{3B\alpha_1^2}{2} - \frac{3B\beta_1^2}{2}\ ,\ S_{14}(0) = \frac{3B_0^3}{2}\ ;\ \ S_{15}(\tau) = 3B\alpha_1\beta_1\ , \tag{5.18d}$$
$$S_{15}(0) = 0$$

$$S_{16}(\tau) = \frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^2}{4},\ \ S_{16}(0) = -\frac{B_0^3}{4}\ ; S_{16}(0) = -\frac{B_0^3}{4}; \tag{5.18e}$$

$$S_{17}(\tau) = \frac{3\alpha_1^2\beta_1}{4} - \frac{\beta_1^3}{4}\ ,\ \ S_{17}(0) = 0 \tag{5.18f}$$

Meanwhile, we also have

$$B'(0) = \frac{B_0 f'(0)}{1-\lambda},\ \ B''(0) = \frac{B_0 F_0}{(1-\lambda)^3}\ ,\ \ F_0 = (1-\lambda)f''(0) + 2\lambda(f'(0))^2 \tag{5.19a}$$

From (5.11b) or (5.13a), we get

$$\alpha_1'(0) = -\frac{B_0^2 f'(0)}{4}\ ,\ \ \left.\left(\frac{\lambda f'\alpha_1}{4(1-\lambda f)}\right)'\right|_{\tau=0} = -\frac{B_0^2 F_1}{16(1-\lambda)};\ \ F_1 = 5\lambda(f'(0))^2 + 4(1-\lambda)f''(0) \tag{5.19b}$$

We also get

$$\alpha_1''(0) = \frac{B_0^2 F_2}{16(1-\lambda)^{\frac{3}{2}}},\ \ F_2 = \left(16 - \frac{(1-\lambda)^{\frac{1}{2}} F_1}{B_0}\right) \tag{5.19c}$$

From (5.21a), we get

$$\beta_1'(0) = -\frac{B_0^2}{(1-\lambda)^{\frac{3}{2}}}\ ,\ \ \beta_1''(0) = -\frac{B_0^3 f'(0) F_3}{4(1-\lambda)^{\frac{3}{2}}},\ F_3 = \left[1 + \frac{1}{B_0}\left\{4\lambda + (1-\lambda)^{\frac{1}{2}}(\lambda+4)\right\}\right] \tag{5.19d}$$

We next substitute for terms on the right side of (5.8) and, to ensure a uniformly valid solution in terms of t, equate to zero the coefficients of cost and *sint* in the resultant equation and so obtain respectively

$$\beta_2' - \frac{\lambda f'\beta_2}{4(1-\lambda f)} = \psi_1(\tau) + \frac{Ba\alpha_2}{(1-\lambda f)^{\frac{1}{2}}} \quad \text{and} \quad \alpha_2' - \frac{\lambda f'\alpha_2}{4(1-\lambda f)} = \psi_2(\tau) - \frac{Ba\beta_2}{(1-\lambda f)^{\frac{1}{2}}} \tag{5.20a}$$

where

$$\psi_1(\tau) = \frac{(1-\lambda\,\mathrm{f})^{\frac{1}{2}}}{2}\left[-\frac{\alpha_1''}{1\text{-}\lambda\,\mathrm{f}} + \frac{2\,aS_5}{1\text{-}\lambda\,\mathrm{f}} + \frac{bS_{12}}{1\text{-}\lambda\,\mathrm{f}}\right];$$

$$\psi_2(\tau) = \frac{(1-\lambda\,\mathrm{f})^{\frac{1}{2}}}{2}\left[\frac{\beta_1''}{1\text{-}\lambda\,\mathrm{f}} - \frac{2\,aS_6}{1\text{-}\lambda\,\mathrm{f}} - \frac{bS_{13}}{1\text{-}\lambda\,\mathrm{f}}\right] \quad (5.20b)$$

If we multiply the first of (5.20a) by $\beta_2$ and the second by $\alpha_2$ and add followed by simplification, we obtain

$$(\alpha_2+\beta_2)' - \frac{\lambda\,\mathrm{f}'(\alpha_2+\beta_2)}{2(1-\lambda\,\mathrm{f})} = 2(\psi_1\beta_2+\psi_2\alpha_2) \quad (5.21a)$$

On solving (5.21a), subject to (5.16bb), we get

$$(\alpha_2+\beta_2) = (1-\lambda\,\mathrm{f})^{-\frac{1}{2}}\left[2\int_0^\tau(\psi_1\beta_2+\psi_2\alpha_2)(1-\lambda\,\mathrm{f(s)})^{\frac{1}{2}} + (1-\lambda)^{\frac{1}{2}}\left\{(\alpha_1(0))^2+(\beta_1(0))^2\right\}\right] \quad (5.21b)$$

We shall not embark on a full determination of $\alpha_2$ and $\beta_2$ as every detailed information about them can easily be obtained from either (5.20a) or (5.21b) .The remaining equation in the substitution into (5.8) are

$$L\zeta^{(3)} = S_{18} + S_{19}\cos 2t + S_{20}\sin 2t + S_{21}\cos 3t + S_{22}\sin 3t \quad (5.22)$$

$$\zeta^{(3)}(0,0) = 0\ ,\ \zeta_t^{(3)}(0,0) + (1-\lambda)^{-\frac{1}{2}}\zeta_\tau^{(2)}(0,0) = 0 \quad (5.23)$$

where

$$S_{18}(\tau) = \frac{2aS_4}{1-\lambda f} + \frac{bS_{11}}{1-\lambda f} - \frac{B''}{1-\lambda f},\ \mathrm{S}_{18}(0) = \frac{13a^2B_0^3}{3(1-\lambda)^2} - \frac{B_0F_0}{(1-\lambda)^3} + \frac{5bB_0^3}{2(1-\lambda)} \quad (5.24a)$$

$$S_{19}(\tau) = \frac{4a(1-\lambda f)^{-\frac{1}{2}}S_3'}{3} - \frac{\lambda f' aS_3}{3(1-\lambda f)^{\frac{3}{2}}} + \frac{2aS_7}{(1-\lambda f)} + \frac{bS_{14}}{(1-\lambda f)} \quad (5.24b)$$

$$S_{19}(0) = \frac{a^2B_0^3(7-3\lambda)}{3(1-\lambda)^2} + \frac{3bB_0^3}{2(1-\lambda)} \quad (5.24c)$$

$$S_{20}(\tau) = -\frac{4a(1-\lambda f)^{-\frac{1}{2}}S_2'}{3} + \frac{\lambda f' aS_2}{3(1-\lambda f)^{\frac{3}{2}}} + \frac{2aS_8}{(1-\lambda f)} + \frac{bS_{15}}{(1-\lambda f)} \quad (5.24d$$

$$S_{21}(\tau) = \frac{2aS_9}{(1-\lambda f)} + \frac{bS_{16}}{(1-\lambda f)},\ \mathrm{S}_{21}(0) = \frac{a^2B_0^3}{(1-\lambda)^2} - \frac{bB_0^3}{4(1-\lambda)} \quad (5.24e)$$

$$S_{22}(\tau) = \frac{2aS_{10}}{(1-\lambda f)} + \frac{bS_{17}}{(1-\lambda f)},\ \ \mathrm{S}_{22}(0) = 0 \quad (5.26f)$$

The solution of (5.22, 5.23) is

$$\zeta^{(3)}(t,\tau) = \alpha_3(\tau)\cos t + \beta_3(\tau)\sin t + S_{18} - \frac{1}{3}(S_{19}\cos 2t + S_{20}\sin 2t) - \frac{1}{8}(S_{21}\cos 3t + S_{22}\sin 3t) \quad (5.25a)$$

$$\alpha_3(0) = -\frac{a^2B_0^3(247+33\lambda)}{72(1-\lambda)^2} + \frac{B_0F_0}{(1-\lambda)^3} - \frac{65\,bB_0^3}{32(1-\lambda)} \quad (5.25b)$$

$$\beta_3(0) - \frac{2}{3}S_{20}(0) - \frac{3}{8}S_{22}(0) + (1-\lambda)^{-\frac{1}{2}}\left[\alpha_2'(0) + a\left\{S_1'(0) - \frac{1}{3}S_2'(0)\right\}\right] = 0 \quad (5.25c)$$

Meanwhile, from (5.29a) or (5.21b), we have

$$\beta_{20}'(0) = 0\ ,\ \alpha_{20}'(0) = \frac{B_0^3 f'(0)F_4}{24(1-\lambda)^3},\ \ \mathrm{F}_4 = a(1-\lambda)\left\{\frac{6(4-\lambda)}{B_0} - 8\right\} - 3F_3 \quad (5.25d)$$

As a summary so far, we write

$$\xi(t,\tau)=\in\zeta^{(1)}(t,\tau)+\in^2\zeta^{(2)}(t,\tau)+\in^3\zeta^{(3)}(t,\tau)+O(\in^4) \qquad (5.26)$$

## 6.0 Maximum Displacement

The condition for maximum displacement is

$$\xi_t+(1-\lambda f)^{-\frac{1}{2}}\xi_\tau=0 \qquad (6.1)$$

We shall let $t_a$ , $\tilde{t}_a$ and $\tau_a$ be the critical values of $t$, $\tilde{t}$ and $\tau$ respectively at maximum displacement, and now assume the following series

$$t_a=t_0+\in t_1+\in^2 t_2+\in^3 t_3+\cdots; \quad \tilde{t}_a=\tilde{t}_0+\in\tilde{t}_1+\in^2\tilde{t}_2+\in^3\tilde{t}_3+\cdots \qquad (6.2a)$$

$$\tau_a=\in t_a=\in\left[t_0+\in t_1+\in^2 t_2+\in^3 t_3+\cdots\right] \qquad (6.2b)$$

We shall now determine some of the terms in (6.26a,b) .We simplify (6.1), using (6.2a,b) and thereafter, equate to zero the coefficients of $\in$ and $\in^2$ and get respectively

$$-\alpha_1(0)\sin t_0+\beta_1(0)\sin t_0=0 \qquad (6.3a)$$

and

$$\Big[-\alpha_1(0)t_1\cos t_0-\alpha_1'(0)\tilde{t}_0\sin t_0+\beta_1'(0)\tilde{t}_0\cos t_0-\alpha_2(0)\sin t_0+\beta_2(0)\cos t_0$$
$$+\frac{2}{3}S_2(0)\sin 2t_0-\frac{2}{3}S_3(0)\cos 2t_0+(1-\lambda)^{-\frac{1}{2}}\alpha_1'(0)\cos t_0+(1-\lambda)^{-\frac{1}{2}}\beta_1'(0)\sin t_0 \qquad (6.3b)$$
$$+(1-\lambda)^{-\frac{1}{2}}B'(0)\Big]=0$$

From (6.3a), we have, $t_0=\pi$ (6.4a)

where we have retained only the least nontrivial positive value of $t_0$ .

From (6.3b), we get

$$t_1=\frac{2B_0\tilde{t}_0+f'(0)(4-\lambda)}{2(1-\lambda)^{\frac{3}{2}}} \qquad (6.4b)$$

where $\tilde{t}_0$ is yet to be determined. From the second of (5.3) evaluated at the critical values, we have

$$t_a=\int_0^{\tilde{t}_a}(1-\lambda f(\in s))^{\frac{1}{2}}ds=(1-\lambda)^{\frac{1}{2}}\left[\tilde{t}_a-\frac{\lambda\in f'(0)\tilde{t}_a^{\,2}}{4(1-\lambda)}-\frac{\in^2\tilde{t}_a^{\,3}}{3}\left\{\frac{\lambda f''(0)}{4(1-\lambda)}+\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2(f'(0))^2\right\}\right]+\cdots \qquad (6.5)$$

If we substitute for $t_a$ and $\tilde{t}_a$ into (6.5) from (6.2a, b) and equate the coefficients of O(1) and $\in$ , we get

$$\tilde{t}_0=\frac{\pi}{(1-\lambda)^{\frac{1}{2}}} \quad\text{and}\quad \tilde{t}_1=\frac{\lambda\pi}{(1-\lambda)^{\frac{5}{2}}}+\frac{f'(0)(8+\pi\lambda-2\lambda)}{4(1-\lambda)^2} \qquad (6.6)$$

The maximum displacement $\xi_m$ is now determined by evaluating (5.27) at the critical values $(t_a,\tau_a)$, using (6.2a,b) and (6.6) . Thus we have

$$\xi_m=\in\zeta^{(1)}(t_a,\tau_a)+\in^2\zeta^{(2)}(t_a,\tau_a)+\in^3\zeta^{(3)}(t_a,\tau_a)+\cdots \qquad (6.7)$$

On simplifying, we get

$$\xi_m=2\in B_0+\in^2\left[(B'-\alpha_1')\tilde{t}_0+\left\{a\left(S_1-\frac{S_2}{3}\right)-\alpha_2\right\}\right]\Bigg|_{\tau=0}+\in^3\left[\frac{S_{21}}{8}+(B'-\alpha_1')\tilde{t}_1\right.$$
$$\left.+\frac{1}{2}(B''-\alpha_1'')\tilde{t}_0^{\,2}+\frac{1}{2}\alpha_1t_1^2-\beta_1'\tilde{t}_0t_1-\alpha_2'\tilde{t}_0-\beta_2t_1+a\tilde{t}_0\left(\frac{S_2'}{3}-S_1'\right)+\left\{S_{18}-\frac{1}{3}S_{19}-\alpha_3\right\}\right]\Bigg|_{\tau=0}+\cdots \qquad (6.8)$$

On simplifying (6.7), we get, $\xi_m=\in C_1+\in^2 C_2+\in^3 C_3+\cdots$ (6.9a)

$$C_1 = 2B_0,\ C_2 = \frac{8aB_0^2 Q_1}{3(1-\lambda)},\ Q_1 = 1 + \frac{3f'(0)(4+\lambda)\tilde{t}_1}{32aB_0},\ \ C_3 = \frac{4bB_0^3 Q_2}{(1-\lambda)} \tag{6.9b}$$

$$Q_2 = \left[1 + \frac{1}{b}\left(\frac{4}{1-\lambda}\right)\left\{\left\{B_0^{-2}\left\{\frac{f'(0)(4+\lambda)\tilde{t}_1}{4(1-\lambda)} + \frac{F_0\tilde{t}_0^2}{2(1-\lambda)^3} - \frac{t_1^2}{2} + \frac{f'(0)(4-\lambda)t_1}{4(1-\lambda)^{\frac{3}{2}}} - \frac{2F_0}{(1-\lambda)^3}\right\}\right.\right.\right.$$
$$\left.\left.\left.+ \frac{B_0^{-1}\tilde{t}_0 t_1}{(1-\lambda)^{\frac{3}{2}}} + \left\{\frac{f'(0)(4+3\lambda)a\tilde{t}_0}{2\lambda(1-\lambda)} + \frac{a^2(536+57\lambda)}{72(1-\lambda)^2} - \frac{\tilde{t}_0 F_4 f'(0)}{24(1-\lambda)^3} - \frac{\tilde{t}_0^2 F_2}{32(1-\lambda)^{\frac{3}{2}}}\right\}\right\}\right\}\right] \tag{6.9c}$$

All along, we have used the assumption that an arbitrary function $\mu(t_a, \tau_a)$ has the following expansion (using (6.2a,b)):

$$\mu(t_a, \tau_a) = \mu(t_0, 0) + \in (t_1\mu_t + \tilde{t}_0\mu_\tau)$$
$$+ \in^2 \left[t_2\mu_t + \tilde{t}_1\mu_\tau + \frac{\tilde{t}_0^{\,2}}{2}\mu_{\tau\tau} + \frac{1}{2}t_1^2\mu_{tt} + t_1\tilde{t}_0\mu_{t\tau}\right] + \cdots \tag{6.10}$$

where (6.10) is evaluated at $(t_a, \tau_a) = (t_0, 0)$ .

## 7.0 Dynamic buckling load $\lambda_D$

We remark that the analysis is predicated on the use of (4.4) which we now utilize. As noted in [9], the first thing to do is to reverse the series (6.9a) in the form

$$\in = f_1\xi_m + f_2\xi_m^2 + f_3\xi_m^3 + \cdots \tag{7.1a}$$

By substituting into (7.1a) for $\xi_m$ from (7.1a) and equating the coefficients of powers of $\in$ , we get

$$f_1 = \frac{1}{C_1},\ f_2 = -\frac{C_2}{C_1^3}\ ,\ f_3 = \frac{2C_2^2 - C_1C_3}{C_1^5} \tag{7.1b}$$

The maximization (4.4) is now accomplished through (7.1a), bearing in mind that each $f_i$ , $i = 1,2,3,\cdots$, is a function of the load parameter $\lambda$ .This gives

$$f_1 + 2f_2\xi_{mD} + 3f_3\xi_{mD}^2 \qquad = 0 \tag{7.1c}$$

The solution of (7.1c) is

$$\xi_{mD} = \frac{1}{3f_3}\left\{-f_2 \pm \left(f_2^2 - 3f_1f_3\right)^{\frac{1}{2}}\right\} \tag{7.1d}$$

We now simplify the terms in (7.1b,d) and note that

$$f_2 = -\frac{C_2}{C_1^3} = -\frac{aQ_1}{3\lambda},\ f_3 = \frac{2C_2^2 - C_1C_3}{C_1^5} = -\frac{bQ_2Q_3}{4\lambda},\ Q_3 = 1 - \frac{16a^2Q_1^2}{27b(1-\lambda)Q_2} \tag{7.2a}$$

Similarly, we have

$$\left(f_2^2 - 3f_1f_3\right)^{\frac{1}{2}} = \sqrt{\frac{3C_1C_3 - 5C_2^2}{C_1^6}} = \sqrt{\frac{3C_3}{C_1^5}\left(1 - \frac{5C_2^2}{3C_1C_3}\right)} = \sqrt{\frac{3b}{8}}\,\frac{(1-\lambda)^{\frac{1}{2}}Q_4(\lambda)}{\lambda} \tag{7.2b}$$

where

$$Q_4(\lambda) = \sqrt{1 - \frac{40aQ_1^2}{27b(1-\lambda)Q_2}} \tag{7.2c}$$

Thus, from (7.1d), we get

$$\left\{-f_2 - \left(f_2^2 - 3f_1f_3\right)^{\frac{1}{2}}\right\} = \sqrt{\frac{3bQ_2}{8Q_3^5}}\,\frac{(1-\lambda)^{\frac{1}{2}}Q_5(\lambda)}{\lambda},$$
$$Q_5(\lambda) = Q_4\left[1 - \frac{\sqrt{8}Q_1a}{3Q_4\sqrt{3bQ_2}(1-\lambda)^{\frac{1}{2}}}\right] \tag{7.3}$$

where we have taken only the negative sign in (7.1d), the positive sign having no physical relevance in this instance .Thus , the maximum displacement $\xi_{mD}$ at buckling is obtained from (7.1d) as

$$\xi_{mD} = \frac{2(1-\lambda_D)^{\frac{1}{2}}}{\sqrt{6bQ_2}}\left(\frac{Q_5}{Q_6}\right) \quad (7.4)$$

To determine the dynamic buckling load $\lambda_D$, we multiply (7.1a) by 3 and get

$$3\in = \xi_{mD}\left\{3(f_1 + f_2\xi_{mD}) + 3f_3\xi_{mD}^2\right\} \quad (7.5)$$

To simplify further, we make the term $3f_3\xi_{mD}^3$ the subject in (7.1c) and substitute same into (7.5) and simplify to get

$$3\in = \xi_{mD}(2f_1 + f_2\xi_{mD}) = \frac{2\xi_{mD}}{C_1}\left(1 - \frac{C_2\xi_{mD}}{2C_1^2}\right) \quad (7.6)$$

where (7.6) is evaluated at $\lambda = \lambda_D$. On further simplifying (7.6), we get

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6b}\ \lambda_D}{2}|\in|\sqrt{Q_2}\left(\frac{Q_3}{Q_5}\right)\left\{1 - \frac{2aQ_1}{3\sqrt{6bQ_2}(1-\lambda_D)^{\frac{1}{2}}}\left(\frac{Q_5}{Q_3}\right)\right\}^{-1} \quad (7.7)$$

## 8.0 Analysis of result and conclusion

The result (7.7), which is asymptotic in nature, is implicit in the load parameter $\lambda_D$. It is valid provided $|Q_1| < 1, |Q_2| < 1, |Q_3| < 1$ and $|Q_4| < 1$, among others. It is observed that the dynamic buckling load $\lambda_D$ is independent of any specific form of the slowly varying load function $f(\in \tilde{t})$ provided condition (2.6) is satisfied. However $\lambda_D$ depends, as far as the load function $f(\in \tilde{t})$ is concerned, on $f'(0)$ which is the derivative of $f(\in \tilde{t})$ evaluated at the initial time. Using (3.4), we can easily eliminate the imperfection amplitude $\in$ in (7.7) to get

$$\left(\frac{\lambda_D}{\lambda_s}\right) = \frac{\left[(1-\lambda_D)^{\frac{3}{2}}\left\{1 - \frac{2aQ_1}{3\sqrt{6bQ_2}(1-\lambda_D)^{\frac{1}{2}}}\left(\frac{Q_5}{Q_3}\right)\right\}\left\{\frac{3\sqrt{6bQ_2}}{2}\left(\frac{Q_3}{Q_5}\right)\right\}^{-1}\right]}{\left[2\left\{\left\{\left\{\frac{(1-\lambda_s)}{3} + \frac{a^2}{9b}\right\}\left\{\frac{\sqrt{3b}(1-\lambda_s)^{\frac{1}{2}}R^{\frac{1}{2}}}{3b} - \frac{a}{3b}\right\}\right\}\right\} - \frac{a(1-\lambda_s)}{9b}\right]} \quad (8.1)$$

The implication of (8.1) is that we can determine the dynamic buckling load $\lambda_D$ in the case of a slowly varying load, from a knowledge of the static buckling load $\lambda_s$. This by-passes the labour of repeating the arduous process for different imperfection amplitudes

### Appendix

We now demonstrate the solution of (5.11a,b)

For brevity, we recast the equations thus

$$\beta_1' - \frac{\lambda f'\beta_1}{4(1-\lambda f)} = \frac{\alpha_1 B}{(1-\lambda f)^{\frac{3}{2}}}\ ; \quad \alpha_1' - \frac{\lambda f'\alpha_1}{4(1-\lambda f)} = -\frac{\beta_1 B}{(1-\lambda f)^{\frac{3}{2}}} \quad \text{A1}$$

The initial conditions are

$$\alpha_1(0) = -B_0,\ \alpha_1'(0) = -\frac{B_0^2 f'(0)}{4};\quad \beta_1(0) = 0,\ \beta_1'(0) = -\frac{B_0^2}{(1-\lambda)^{\frac{3}{2}}} \quad \text{A2}$$

We let $\alpha_1 = \gamma e^{\varphi(\tau)},\ \beta_1 = \eta e^{\varphi(\tau)},\quad \theta(\tau) = \frac{\lambda f'}{4(1-\lambda f)},\ \Omega(\tau) = \frac{B}{(1-\lambda f)^{\frac{3}{2}}}$ A3

On substituting A3 into A1 and simplifying, we get

$$\eta(\varphi' - \theta) = \Omega\gamma\ ;\quad \eta\Omega = \gamma(\varphi' - \theta) \quad \text{A4}$$

Thus we have $\varphi' = \theta \pm i\Omega = M_1(\tau) \pm iM_2(\tau)\ ;\ M_1 = \int_0^\tau \theta(s)ds, M_2 = \int_0^\tau \Omega(s)ds$ A5

On substituting from A3, we get

$$M_1 = -\frac{1}{4}\ln(1-\lambda f),\ M_1(0) = -\frac{1}{4}\ln(1-\lambda);\ M_2 = \int_0^{\tau}\frac{B(s)ds}{(1-\lambda f)^{\frac{3}{2}}},\ M_2(0) = 0 \qquad \text{A6}$$

On substituting into A3 for $\varphi(\tau)$, we get

$$\alpha_1(\tau) = \lambda e^{M_1 \pm iM_2} = e^{M_1}\{\gamma_{11}\cos M_2 + \gamma_{12}\sin M_2\} \qquad \text{A7}$$

$$\beta_1(\tau) = \eta e^{M_1 \pm iM_2} = e^{M_1}\{\eta_{11}\cos M_2 + \eta_{12}\sin M_2\} \qquad \text{A8}$$

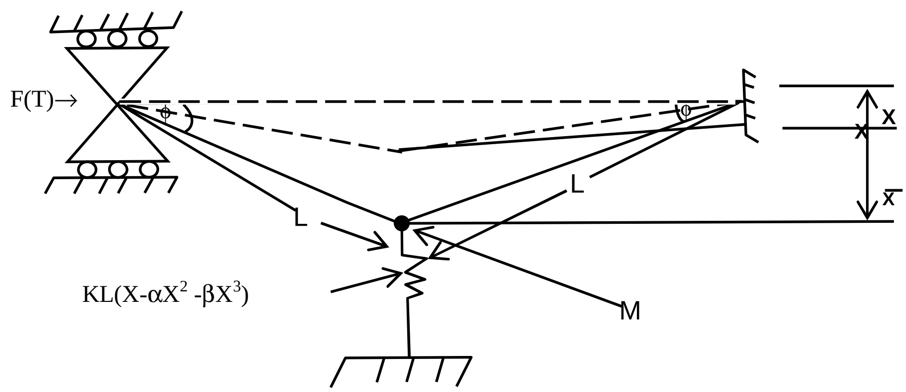
On using the initial conditions for $\alpha_1(\tau)$ from A2, we get

$$\gamma_{11} = -B_0 e^{-M_1(0)},\ \gamma_{12} = \frac{e^{-M_1(0)}}{\Omega(0)}\left\{\theta(0)B_0 - \frac{f'(0)B_0^2}{4}\right\} \qquad \text{A9}$$

Similarly, on using the initial conditions for $\beta_1(\tau)$ from A2, we obtain

$$\eta_{11} = 0,\ \eta_{12} = -\frac{B_0^2 e^{-M_1(0)}}{\Omega(0)(1-\lambda)^{\frac{3}{2}}} \qquad \text{A10}$$

On substituting for the evaluated coefficients in A7 from A9 and in A8 from A10, we obtain the results in (5.13a,b).




**A Simple Quadratic – Cubic Elastic Model Structure**

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