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On a differential subordination of some certain subclass of Univalent function

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Abstract

We generate some results for some particular subclasses of starlike and close-to-convex functions using Briot-Bouquet differential subordination method.

1.0 Introduction

Let A denote the class of function f(z) of the form $f(z)=z+\sum_{k=2}^{\infty}a_k z^k$, which are analytic and univalent in the unit disk U = {z:|z|<1}, normalized by the conditions $f(0) = 0$, $f'(0) = 1$, With S*(α) and *CC*(α) denoting the subclasses of *A* that are, respectively starlike and close-to-anvere of order α, α∈ [-1, 1), see [5]. Also, for two functions *f* (z) and *g* (z) analytic in *U*, we say that the function *f* (z) is subordinate to *g* (z) in U, and write, $f(z) \prec g(z)$ or $f \prec g$, z ∈ U, if there exists a Schwarz function *w*(*z*), analytic in U with $w(0) = 0$ and $|w(z)| < 1$, z ∈ *U*, such that, $f(z)=g(w(z))$, z ∈ U.

In particular, if the function *g*(z) is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

The general theory of differential subordination was introduced in 1981 by Millerr and Mocanu (see 2). The first – order differential subordination

$$p(z)+\frac{zp'(z)}{\beta\, p(z)+\gamma} \prec h(z),\ z\in U,\ \gamma\neq 0,\ \mathrm{Re}\,\gamma\geq 0,$$

h is convex and $p(z)$ regular in U known as the Briot – Bouquet differential subordination was introduced by the same author as a special case of the general theory of differential subordination [see 2]. Some of its properties were studied by the same author in [3, 4].

This first order differential subordination with many interesting applications in the theory of univalent functions was considered by many authors see (3, 4 and 5).

The main purpose of this work is to apply a method based on the Briot-Bouquet differential subordination to derive some subordination results involving some particular classes of starlike and close-to-convex functions for special values of β, γ *and* h.

2.0 Preliminary results:

In order to prove our main results, we shall need the following definitions and lemmas.

***Definition* 2.1**

For a function f(z)∈A, let D^n be the salagean differential operator defined in [1] *as* $D^0 f(z) = f(z)$ $D^1 f(z) = D f(z)\, z f^1(z)$, $D^n f(z) = D[D^{n-1} f(z)] = z[D^{n-1} f(z)]^1$, $z\in U$, $n \geq 1$.

With the above definition, we introduce some classes of A denoted by $S_n^*(\alpha, r)$ and $CC_n(\alpha,r)$ as follows:

***Definition* 2.2**

Let f ∈ A, we say that the function f (z)∈ $S_n^*(\alpha, r)$, *if and only if*

$$\mathrm{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \prec \left(\frac{1+\alpha z}{1+r\,z}\right),\ z\in U,$$

where α ≥ 0, $r\in[-1.1)$, α + r ≥ 0, $n \in N$.

***Definition* 2.3**

Let f (z)∈ A, we say that the function f (z)∈ $CC_n(\alpha, r)$ in respect to the function g(z)∈ $S_n^*(\alpha, r)$ where α ≥ 0, r∈[-1,1), (α+r) ≥ 0 if and only if $\mathrm{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \prec \left(\frac{1+\alpha z}{1+r\ z}\right)$, *z ∈ U, where α ≥ 0, r∈[-1.1), α+r ≥ 0, n∈N.*

***Lemma* 2.1**.[3,4]

Let h(z) be convex in U and Re [βh(z) + r) > 0, z∈U. If P(z) is analytic in U with P(0)=h(0) and P(z) satisfied the Briot-Bouquet differential subordination $p(z)+\frac{zp'(z)}{\beta p(z)+r} \prec h(z)$, *then* $p(z) \prec h(z)$.

***Lemma* 2.2** [2,4]

Let q(z) be convex in U and j(z) be analytic in U with Re[j(z)]>0. If p(z) is analytic in U and p(z) satisfied the differential subordination $p(z)+j(z).zp'(z) \prec q(z)$, *then* $p(z) \prec q(z)$

3.0 Main Results

***Theorem* 3.1**

If F(z)∈ $S_n^*(\alpha, r)$ *with α≥0, r∈[-1,1), then the integral operator f(z) defined by*

$$f(z) = \frac{c}{z^{c+y}}\int_0^z \frac{zF(t)t^{c+y}}{t}\,dt \tag{3.1}$$

z∈U, c,y∈N, is also in $S_n^*(\alpha, r)$.

Proof:

Let *F*(z)∈ $S_n^*(\alpha, r)$, then by definition $\mathrm{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \prec \left(\frac{1+\alpha z}{1+rz}\right)$, z ∈ U, α ≥ 0, $r\in(-1,1)$. By differentiating (3.1) we obtain:

$$cF(z) = zf'(z) + (c+y)f(z) \tag{3.2}$$

By applying the linear operator D^{n+1} we obtain:

$$CD^{n+1}F(z) = D^{n+2}f(z) + D^{n+1}(c+y)f(z) \tag{3.3}$$

Similarly, application of the linear operator D^n yields:

$$CD^{n}F(z) = D^{n+1}f(z) + (c+y)D^{n}f(z) \tag{3.4}$$

Thus,

$$\frac{CD^{n+1}F(z)}{CD^nF(z)} = \frac{D^{n+2}f(z)+(c+y)D^{n+1}f(z)}{D^{n+1}f(z)+(c+y)D^nf(z)}$$

$$= \frac{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\bullet\frac{D^{n+1}f(z)}{D^nf(z)} + \frac{(c+y).D^{n+1}f(z)}{D^nf(z)}}{\frac{D^{n+1}f(z)}{D^nf(z)}+(c+y)} \tag{3.5}$$

By setting

$$\frac{D^{n+1}f(z)}{D^nf(z)} = p(z) \tag{3.6}$$

So that *p* (*z*) has the following series expansion; $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \ldots$,

By differentiating (3.6), we have,

$$zp'(z) = z.\left(\frac{D^{n+1}f(z)}{D^nf(z)}\right)'$$

$$= \frac{z\left[D^nf(z).(D^{n+1}f(z))' - D^{n+1}f(z)(D^nf(z))'\right]}{(D^nf(z))^2}$$

$$= \frac{D^nf(z)D^{n+2}f(z) - (D^{n+1}f(z))^2}{(D^nf(z))^2} \tag{3.7}$$

Also, $$\frac{1}{p(z)}.zp'(z)=\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^{n}f(z)}$$

$$\therefore \quad \frac{D^{n+2}f(z)}{D^{n+1}f(z)}=\frac{1}{p(z)}.zp'(z)+p(z) \tag{3.8}$$

From (3.5), we obtain,

$$\frac{D^{n+1}F(z)}{D^{n}F(z)}=\frac{\left(\frac{zp'(z)}{p(z)}+p(z)\right).p(z)+(c+y).p(z)}{p(z)+(c+y)}$$

$$=p(z)+\frac{zp'(z)}{p(z)+(c+y)} \tag{3.9}$$

Conclusion follow from Lemma (2.1) by considering h(z) to be convex in U with $h(0) = 1$, $Re(c + y) \geq 0$ and thus, $Re(h(z) + (c + y)) > 0$ and with Lemma (2.1), we obtain: $p(z)\prec h(z)$. Thus, $p(z)=\frac{D^{n+1}f(z)}{D^{n}f(z)}\prec h(z)$.

By taking $h(z)=\frac{1+\alpha z}{1+rz}$. Hence, $f(z)=\frac{c}{z^{c+y}}\int_0^z\frac{zF(t)t^{c+y}}{t}dt \in S_n^*(\alpha,r)$, z∈U.

***Theorem* 3.2**

If $F(z)\in CC_n(\alpha,r)$, in respect to the function $g(z)\in S_n^*(\alpha,r)$ with $\alpha\geq 0$, $r\in[-1,1)$, then the integral operator f(z) defined by

$$f(z)=\frac{c}{z^{c+y}}\int_0^z\frac{zF(t)t^{c+y}}{t}dt\ ,\ z\in U,\ c,y\in N \tag{3.10a}$$

is also in $CC_n(\alpha,r)$ in respect to the function $S_n^*(\alpha,r)$,

$$g(z)=\frac{c}{z^{c+y}}\int_0^z\frac{zG(t)t^{c+y}}{t}dt\ z\in U,\ c,\ y\in N\neq 0 \tag{3.10b}$$

Proof

Let $F(z)\in CC_n(\alpha,r)$, in respect to $G(z)\in S_n^*(\alpha,r)$, then by definition

$$\mathrm{Re}\left(\frac{D^{N+1}F(z)}{D^{n}G(z)}\right)\prec\frac{1+\alpha z}{1+rz},\ z\in U,\ \alpha\geq 0,\ r\in[-1,1)$$

By differentiating (3.10) we obtain: $cF(z)=zf'(z)+(c+y)f(z)$ and $G(z)=zg'(z)+(c+y)g(z)$

By the application of the linear operator D^{n+1}, we obtain, $CD^{n+1}F(z)=D^{n+2}f'(z)+(c+y)D^{n+1}f(z)$.

Similarly, the application of the linear operator D^n, we obtain, $CD^{n}G(z)=D^{n+1}g(z)+(c+y)D^{n}g(z)$.

Thus, simple calculation, shows that,

$$\frac{D^{n+1}F(z)}{D^{n}G(z)}=\frac{\frac{D^{n+2}f(z)}{D^{n+1}g(z)}\bullet\frac{D^{n+1}g(z)}{D^{n}g(z)}+\frac{(c+y).D^{n+1}f(z)}{D^{n}g(z)}}{\frac{D^{n+1}g(z)}{D^{n}g(z)}+(c+y)} \tag{3.11}$$

By setting $\frac{D^{n+1}f(z)}{D^{n}g(z)}=p(z)$, and $\frac{D^{n+1}g(z)}{D^{n}g(z)}=k(z)$ (3.12)

By differentiating (3.12) logarithmically, we obtain,

$$zp'(z)=z.\left(\frac{D^{n+1}f(z)}{D^{n}g(z)}\right)'=\frac{D^{n+2}f(z).D^{n}g(z)-D^{n+1}f(z).D^{n+1}g(z)}{\left(D^{n}g(z)\right)^{2}}$$

and $\frac{1}{k(z)}.zp'(z) = \frac{D^{n+2}f(z)}{D^{n+1}g(z)} - \frac{D^{n+1}f(z)}{D^{n}g(z)}$, which lead us to,

$$\frac{D^{n+2}f(z)}{D^{n+1}g(z)} = \frac{1}{k(z)}.zp'(z) + p(z) \qquad (3.13)$$

Thus, from (3.11),

$$\frac{D^{n+1}F(z)}{D^{n}G(z)} = \frac{k(z).\left(\frac{1}{k(z)}.zp'(z) + p(z)\right) + (c+y).p(z)}{k(z)+(c+y)} = p(z) + \frac{zp'(z)}{k(z)+(c+y)} \qquad (3.14)$$

The conclusion follows form Lemma (2.2), by taking *q(z)* to be convex in *U*, then $\frac{D^{n+1}F(z)}{D^{n}G(z)}$ $= p(z) + \frac{zp'(z)}{k(z)+(c+y)} \prec q(z)$, where from the condition of the theorem, we have *Rek(z)* > 0 and *Re(c + y) ≥ 0*, thus, $\operatorname{Re}\frac{1}{k(z)+(c+y)} > 0$. With this condition and from Lemma (2.2) and taking $j(z) = \frac{1}{k(z)+(c+y)}$, we obtain, $p(z) \prec q(z)$. From here it follows that, if, $\operatorname{Re}\left(\frac{D^{n+1}F(z)}{D^{n}G(z)}\right) \prec \frac{1+\alpha z}{1+rz}$ then $p(z) = \frac{D^{n+1}f(z)}{D^{n}g(z)} \prec \frac{1+\alpha z}{1+rz}$. Taking *q*(*z*) to be $\frac{1+\alpha z}{1+rz}$. Hence, f$(z) \in CC_n(\alpha,r)$, in respect to $g(z) \in S_n^*(\alpha,r)$, α ≥ 0, $r \in [1,1]$.

2.0 Conclusion

In this paper we are able to generate some subordination results for some particular subclasses of univalent functions (mainly, the starlike and close – to – convex functions) via a method based upon a special; case of differential subordination known as Briot – Bouquet differential subordination for special values of β, γ and *h.*

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