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# Some remarks on certain Bazilevic functions

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**Abstract**

**In this note we give some sufficient conditions for an analytic function $f(z)$ normalized by $f'(0) = 1$ to belong to certain subfamilies of the class of Bazilevic functions. In earlier works, the closure property of many classes of functions under the Bernardi integral have been considered. The converse of this problem is also considered here.**

***Keywords**:* Bazilevic functions, analytic and univalent functions

## 1.0 Introduction

Let $C$ be the complex plane. Denote by $A$ the class of functions:

$$f(z) = z + a_2 z^2 + \ldots \tag{1.1}$$

which are analytic in the unit disk $E = \{z: |z| < 1\}$. Let $S$ be the subclass of $A$ consisting only of univalent functions in $E$. Bazilevic [4] isolated a subclass of $S$, which consists of functions defined by the integral

$$f(z) = \left\{ \frac{\alpha}{1+\xi^2} \int_0^z [p(v) - i\xi] v^{-\left(1+\frac{i\alpha\xi}{1+\xi^2}\right)} g(v)^{\left(\frac{\alpha}{1+\xi^2}\right)} dv \right\}^{\frac{1+i\xi}{\alpha}} \tag{1.2}$$

where $p \in P$ (consisting of analytic functions $p(z)$ which have positive real part in $E$ and normalized by $p(0) = 1$) and $g \in S^*$ (the subclass of $S$ satisfying Re $zg'(z)/g(z) > 0$, that is,. starlike in $E$). The numbers $\alpha > 0$ and $\xi$ are real and all powers are meant as principal determinations only. The family of functions (1.2), which we denote by $B(\alpha,\xi,p,g)$ is known as family of Bazilevic functions. He proved that every Bazilevic function is univalent in $E$. Apart from this, very little is known regarding the family as a whole. However, with some simplifications, it has been possible to understand and investigate the family. Indeed it is easy to verify that with special choices of the parameters $\alpha$ and $\xi$, and the function $g(z)$, the family $B(\alpha,\xi,p,g)$ comes down to some well-known subclasses of univalent functions. For instance, if we put $\xi = 0$, we have

$$f(z) = \left\{ \alpha \int_0^z \frac{p(v)}{v} g(v)^\alpha \, dv \right\}^{\frac{1}{\alpha}} \tag{1.3}$$

On differentiation, the expression (1.3) yields

$$\frac{zf'(z) f(z)^{\alpha-1}}{g(z)^\alpha} = p(z), \qquad z \in E \tag{1.4}$$

that is,

$$\text{Re}\, \frac{zf'(z) f(z)^{\alpha-1}}{g(z)^\alpha} > 0, \qquad z \in E \tag{1.5}$$

The subclasses of Bazilevic functions satisfying (1.5) are called Bazilevic functions of type $\alpha$ and are denoted by $B(\alpha)$ (see [8]). Noonan [6] gave a plausible description of functions of the class $B(\alpha)$

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as those functions in $S$ for which for each $r < 1$, the tangent to the curve $U_\alpha(r) = \{f(re^{i\theta})^\alpha: \ 0 \le \theta < 2\pi\}$ never turns back on itself as much as $\pi$ radian.

With specific choices of the associated parameters the class $B(\alpha)$ reduces to the well-known families of close-to-convex, starlike and convex functions. Furthermore, if $g(z) = z$ in (1.5) we have the family $B_1(\alpha)$ [8] of functions satisfying:

$$\operatorname{Re} \frac{zf'(z)f(z)^{\alpha-1}}{z^\alpha} > 0, \qquad z \in E \tag{1.6}$$

Abdulhalim [1] introduced and studied a generalization of functions satisfying (1.6) as follows:

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{z^\alpha} > 0, \qquad z \in E \tag{1.7}$$

where the operator $D^n$ is the well-known Salagean derivative operator defined by the relations $D^0 f(z) = f(z)$ and $D^n f(z) = z[D^{n-1}f(z)]'$ [1]. He denoted this class of functions by $B_n(\alpha)$. It is immediately obvious that Abdulhalim's generalization has extraneously included analytic functions satisfying:

$$\operatorname{Re} \frac{f(z)^\alpha}{z^\alpha} > 0, \qquad z \in E \tag{1.8}$$

which are largely non-univalent in the unit disk. The case $\alpha = 1$ here, coincides with the class of functions studied by Yamaguchi [9].

Opoola [7] further generalized functions defined by the geometric condition (1.7) by choosing a real number $\beta$ $(0 \le \beta < 1)$ such that:

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta, \qquad z \in E \tag{1.9}$$

(see [2,7]).

In this article we provide sufficient conditions for an analytic function $f \in A$ to belong to certain subfamilies of the class of Bazilevic functions. In section 2 we state some preliminary lemmas. We prove the main results in section 3.

## 2.0 Preliminary lemmas

***Definition* 2.1.**

Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$. Define $\Psi$ as the set of functions $\psi(u,v): C \times C \to C$ satisfying:

(a) $\psi(u,v)$ is continuous in a domain $\Omega$ of $C \times C$

(b) $(1,0) \in \Omega$ and $\operatorname{Re}\psi(1,0) > 0$

(c) $\operatorname{Re}\psi(u_2 i, v_1) \le 0$ when $(u_2 i, v_1) \in \Omega$ and $2v_1 \le -(1+u_2^2)$.

The following two examples of the set $\Psi$ add to those mentioned in [2, 5].

(i) $\psi_1(u,v) = v/\xi u$, $\xi > 0$ is real and $\Omega = [C-\{0\}]\times C$

(ii) $\psi_2(u,v) = \frac{1}{2} + v/(\xi(1+u))$, $0 < \xi \le 1$ and $\Omega = [C-\{-1\}]\times C$.

***Definition* 2.2**.

Let $\psi \in \Psi$ with corresponding domain $\Omega$. Define $P(\Psi)$ as the set of functions $p(z)$ given as $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ which are regular in $E$ and satisfy:

(i) $(p(z), zp'(z)) \in \Omega$

(ii) $\operatorname{Re}\psi(p(z), zp'(z)) > 0$ when $z \in E$.

The above definitions are abridged forms of the more general concepts discussed in [2].

***Lemma* 2.3**. [2]

*Let* $p \in P(\Psi)$. *Then* $\operatorname{Re} p(z) > 0$.

***Lemma*** **2.4**. [3]

*Let $f \in A$, and $\alpha > 0$ be real. If $D^{n+1}f(z)^{\alpha}/D^n f(z)^{\alpha}$ takes a value which is independent of n, then*

$$\frac{D^{n+1}f(z)^{\alpha}}{D^n f(z)^{\alpha}} = \alpha \frac{D^{n+1}f(z)}{D^n f(z)}$$

**Lemma 2.5. [8]**

*Let $p \in P$. Then* $|zp'(z)| \le \frac{2r}{1-r^2}\operatorname{Re} p(z), \qquad |z| = r$

*The result is sharp. Equality holds for the function* $p(z) = (1 + z)/(1 - z)$.

## 3.0 Main Results

**Theorem 3.1**

*Let $f \in A$. If $\alpha > 0$ and* $\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right) > 0$ *then $f \in B_n(\alpha)$.*

**Proof.**

Let $p(z) = \frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}}$. Then we have $\frac{1}{\alpha}\frac{D^{n+1}f(z)^{\alpha}}{D^n f(z)^{\alpha}} - 1 = \frac{zp'(z)}{\alpha p(z)}$. Thus applying Lemma 2.4 we obtain $\frac{D^{n+1}f(z)}{D^n f(z)} - 1 = \frac{zp'(z)}{\alpha p(z)} = \psi(p(z), zp'(z))$ where $\psi = \psi_1(u,v) = \xi v/u$ (with $\xi = 1/\alpha$) belongs to $\Psi$. Hence, by Lemma 2.3, we have the implication that $\operatorname{Re}\psi(p(z), zp'(z)) > 0 \Rightarrow \operatorname{Re} p(z) > 0$, that is, $\operatorname{Re}\frac{D^n f(z)^{\alpha}}{z^{\alpha}} > 0$, that is, $f(z)$ belongs to the class $B_n(\alpha)$..

***Corollary*** **3.2**.

*Let $f \in A$. If $\alpha > 0$, then*

(*a*) $\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) > 0 \Rightarrow \operatorname{Re}\frac{f(z)^{\alpha}}{z^{\alpha}} > 0$

(*b*) $\operatorname{Re}\frac{zf''(z)}{f'(z)} > 0 \Rightarrow \operatorname{Re}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0$

In particular if $\alpha = 1$, we have

***Corollary*** **3.3.**

*Let $f \in A$. Then*

(*a*) $\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) > 0 \Rightarrow \operatorname{Re}\frac{f(z)}{z} > 0$

(*b*) $\operatorname{Re}\frac{zf''(z)}{f'(z)} > 0 \Rightarrow \operatorname{Re} f'(z) > 0$

***Remark*** **3.4.**

In fact for $0 \le \beta < 1$, $\psi_1(u,v) = v/\xi u$, with $\xi > 0$ real and $\Omega = [C\text{-}\{0\}]\times C$ is found to be contained in the set $\Psi_{\beta}$, $0 \le \beta < 1$, defined in [2]. Thus the above theorem and corollaries can be extended easily to the class $T_n^{\alpha}(\beta)$.

***Theorem*** **3.5.**

*Let $f \in A$. If $0 < \alpha \le 1$, then*

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \frac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \frac{1}{2}$$

**Proof**

Let $p(z) = 2\dfrac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1$ then, $\dfrac{D^{n+1} f(z)^\alpha}{\alpha D^n f(z)^\alpha} = 1 + \dfrac{zp'(z)}{\alpha(1+p(z))}$

Appealing to Lemma 2.4, we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{1}{2} = \frac{1}{2} + \frac{zp'(z)}{\alpha(1+p(z))} = \psi(p(z), zp'(z))$$

where $\psi_2(u,v) = \frac{1}{2} + v/(\xi(1+u))$ with $\xi = \alpha$ and $\Omega = [C\text{-}\{-1\}]\times C$. Therefore, by Lemma 2.3, we have the implication that $\operatorname{Re}\psi(p(z), zp'(z)) > 0 \Rightarrow \operatorname{Re} p(z) > 0$, that is,

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \frac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \frac{1}{2}.$$

***Corollary* 3.6.**

*Let $f \in A$. If $0 < \alpha \le 1$, then*

(*a*) $\operatorname{Re}\dfrac{zf'(z)}{f(z)} > \dfrac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\dfrac{f(z)^\alpha}{z^\alpha} > \dfrac{1}{2}$

(*b*) $\operatorname{Re}\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \dfrac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\dfrac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > \dfrac{1}{2}$

Furthermore, if $\alpha = 1$ we have

***Corollary* 3.7.**

*Let $f \in A$. Then*

(*a*) $\operatorname{Re}\dfrac{zf'(z)}{f(z)} > \dfrac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\dfrac{f(z)}{z} > \dfrac{1}{2}$

(*b*) $\operatorname{Re}\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \dfrac{1}{2} \quad \Rightarrow \quad \operatorname{Re} f'(z) > \dfrac{1}{2}$

***Remark* 3.8**.

The first part of Corollary 3.7 above appeared in [5, Theorem 9(ii)].

In many earlier works, the Bernardi integral operator

$$F(z)^\alpha = \frac{\alpha + c}{z^c}\int_0^z t^{c-1} f(t)^\alpha dt, \quad \alpha + c > 0 \tag{3.1}$$

has received much attention (see [1-2,7-8]). The problem is to study the closure property of classes of functions under the operator. The converse problem of determining the largest circle in which $f(z)$ defined by the integral (3.1) belongs to a certain class of function, given that the function $F(z)$ is a member of the class, has also been considered by some authors (see [8]). Our concern here is to generalize and improve those earlier results. Our result is stated as follows.

***Theorem* 3.9.**

*Let* $F(z) \in T_n^\alpha(\beta)$ *and* $\alpha + c > 0$. *Then f given by* (3.1) *is in* $T_n^\alpha(\beta)$ *provided* $|z| < r_0(\alpha,c)$ *where* $r_0(\alpha,c)$ *is given by*

$$r_0(\alpha, c) = \frac{(1+(\alpha+c)^2)^{\frac{1}{2}} - 1}{(\alpha+c)} \tag{3.2}$$

*The result is sharp.*

**Proof**.

From (3.1), we get

$$(\alpha + c) f(z)^\alpha = DF(z)^\alpha + cF(z)^\alpha \tag{3.3}$$

Hence we have

$$(\alpha + c) \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{D^{n+1} F(z)^\alpha}{\alpha^n z^\alpha} + c \frac{D^n F(z)^\alpha}{\alpha^n z^\alpha} \tag{3.4}$$

Since $F(z) \in T_n^\alpha(\beta)$, there exists $p \in P$ such that

$$\frac{D^n F(z)^\alpha}{\alpha^n z^\alpha} = \beta + (1-\beta) p(z) \tag{3.5}$$

Therefore,

$$\frac{D^{n+1} F(z)^\alpha}{\alpha^n z^\alpha} = \alpha(\beta + (1-\beta) p(z)) + (1-\beta) z p'(z) \tag{3.6}$$

Using (3.5) and (3.6) in (3.4), we have

$$(\alpha + c)\left( \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta \right) = (1-\beta)[(\alpha + c) p(z) + z p'(z)] \tag{3.7}$$

Applying Lemma 2.5 on (3.7), we obtain

$$(\alpha + c)\left( \operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta \right) = (1-\beta)\left( \frac{(\alpha + c)(1 - r^2) - 2r}{1 - r^2} \right) \operatorname{Re}\ p(z) \tag{3.8}$$

which implies $\operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta$ , , provided $|z| < r_0(\alpha, c)$.

The functions defined by $\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1 + (1 - 2\beta) z}{1 - z}$ show that the result is sharp. This completes the proof.

***Remark* 3.10.**

We improve Singh's result [8, Theorem 4′] by setting $n = 1$ and $\beta = 0$. The parameter $\alpha$ and $c$ may not necessarily be integers and $c$ may be as small as 'almost' $-\alpha$.

## 4.0 Conclusion

In this study, we have provided a sufficient condition for analytic functions normalized by $f(0) = 0$ and $f'(0) = 1$ to belong to the subclasses, $B_n(\alpha)$, which are known to consist on analytic and univalent functions having logarithmic growth in the unit disk [1]. Also we proved another sufficient condition for such analytic functions to be of order $\frac{1}{2} - B_n(\alpha)$ functions. For $n - 1$, both results are sufficient univalence conditions in the unit disk. Furthermore, we solved the radius problem associated with the well-known Bernardi integral for a more general class of functions, that is, the class $T_n^\alpha(\beta)$.

The results of this work generalize and extend many known ones as we have noted in the corollaries and remarks following each of them.

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