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On finitely many fixed points

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Abstract

Let C be the finite union of closed convex sets in a complete metrisable locally convex space. If f: C → C with $\overline{f(C)}$ compact, then f can be approximated by a map g: C → C which has only a finite number of fixed points. This result, which is a generalization of the result of Baillon and Rallis, is proved in this paper.

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1.0 Introduction

In the last fifty years or so, the theory of fixed points has been a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as biology, Chemistry, Engineering and Physics. It touches on many areas of Mathematics including general topology, algebraic topology, nonlinear functional analysis, and ordinary and partial differential equations and also serves as a useful tool in applied Mathematics. More importantly, fixed point theory method is an essential tool in the study of existence and approximation of solutions of ordinary and partial differential equations.

2.0 Preliminaries

Let C be a subset of a metric space (X, d). We say that the C satisfies Fix-Finite Approximation Property (FFAP) if for any family $\mathscr{F}$ of maps from C to C, and for every $f \in \mathscr{F}$ and all $\varepsilon > 0$, there is a $g \in \mathscr{F}$ such that $d(f(x),g(x)) < \varepsilon$ for all $x \in C$ and g has only a finite number of fixed points [14]. H. Hopf in [7] proved by a special construction that a finite polyhedron which is connected and which its dimension is greater than 1 satisfies FFAP. H.Schirmer [13] extended this result to any continuous n-valued multifunction. J. B. Baillon and N. E. Rallis [1] showed that any finite union of closed convex subsets of a Banach space satisfies the FFAP for any compact self maps (i.e. continuous maps whose closure of its range is compact). A Stouti in [14] also studied the FFAP in normed linear spaces. In this paper, we showed that Baillon and Rallies result can be generalized to locally convex spaces.

We recall the following definitions used in the proof of our Theorem.

***Definition* 2.1** [14]
Let X be a topological space and (Y, d) a metric space. A homotopy h_t: $X \to Y$, $0 \leq t \leq 1$, is said to be ε-homotopy if $\sup\{d(h_t(x), h_{t'}(x))$: $t, t' \in [0, 1]\} < \varepsilon\}$ for all $x \in X$.

***Definition* 2.2** [4]
Let Y be a metric space, then Y is said to be an Absolute Neighbourhood Retract (ANR) if for any nonempty closed subset A of an arbitrary metric space X and for any continuous map f: $A \to Y$, there exists an open subset U of X containing A and a continuous map g: U→Y which is an extension of f, i.e. $g(x) = f(x)$ for all $x \in X$.

In [4], Dugundji established the homotopy extension for ANRs. Finite union of closed convex set in a metric space are examples of ANR's [4, 5].

***Definition* 2.3** [11]

A locally convex linear topological space is a linear topological space which has a base of convex neighbourhoods of the origin. If the intersection of the neighbourhoods of the origin is zero, then we say that the locally convex space is Haudsdorff. If the neighbourhood of the origin is countable and X is Hausdorff, then X is metrisable.

All normed linear spaces are locally convex spaces. For more information on the theory of locally convex spaces, see [8-9], [11], and [12].

***Definition* 2.4** [5, P411]

Let A be any subset of a vector space X. The convex hull $H(A)$ of A is the intersection of all convex sets containing A.

The following result is well known.

***Theorem* 2.1** (A. Tychonoff)

Let $\boldsymbol{X}$ be a locally convex linear topological space and let C be a compact convex subset of X. Then each continuous $f: C \to C$ has a fixed point.

Fixed Point Theory is not necessarily locally convex spaces as discussed in [6].

Tychonoff fixed point Theorem is not immediately applicable in analysis because the required compactness of the domain is difficult to get in practice. The following which was proved by J. Schauder when X is a Banach space is more practicable.

We denote the intersection of all closed convex sets containing B in a locally convex space as $\hat{B}$. $\hat{B}$ is compact whenever X is complete and B is compact [11, p60, Corollary]

***Theorem* 2.2**

Let C be a closed convex subset of a complete locally convex space X and $f: C \to C$ be continuous. If $\overline{f(C)}$ is compact, then f has a fixed point.

Proof

Since $\overline{f(C)}$ is compact, we have that $f(\hat{C})$ is compact by [11, p60, Corollary]. Since C is closed and convex and $\overline{f(C)} \subset \overline{C}$ = C, it follows that $f(\hat{C}) \subset \hat{C}$ = C. Thus $f \mid f(\hat{C}) : f(\hat{C}) \to f(\hat{C})$ and thus by Tychonoff's Theorem, f has a fixed point. Since a Banach space is a complete locally convex space, then we have the following well known result due to J. Schauder.

***Corollary* 2.1** (Tychonoff-Schauder Theorem)

Let C be a closed convex subset of a Banach space X and $f: C \to C$ be continuous. If $\overline{f(C)}$ is compact, then f has a fixed point.

While the Tychonoff -Schauder Theorem is well known, Theorem B is a new result. We shall now state the following theorems which is fundamental to our results in this paper.

***Theorem* 2 3** [11, Chap. 1, Theorem 4].

The topology of a metrisable locally convex space can always be defined by a metric $d(x, y) = F(x-y)$, which is invariant under translation, where $F(x) = \sum_{n=1}^{n=\infty} 2^{-n} \min\{ p_n(x), 1\}$ and $\{p_n\}$ is the set of seminorms describing the locally convex topology.

It should be observed that if X is a normed linear space, then F satisfies the triangle inequality and will also be a norm. It is also easy to see that $F(x) = 0$ implies that x = 0 for any $x \in X$. It is also easy to prove that if X is a metrisable locally convex space, $F(\lambda x) \leq F(x)$ for any x whenever $0 \leq \lambda \leq 1$. Henceforth F will denote the function as defined above.

***Theorem* 2.4** [4]

Let X be a metrisable space and Y a metric ANR. For $\varepsilon > 0$, there exists $\delta > 0$ such that for any two maps $f,g: X \to Y$ such that $d(f(x),g(x)) < \delta$ for all $x \in X$ and δ-homotopy $j_t: A \to Y$ where A is a closed subspace of X and $j_o = f|A$ and $j_1 = g|A$, there is an ε-homotopy $h_t: X \to Y$ such that $h_o = f$ and $h_1 = g$ and $h_t|A = j_t$ for every $t \in I$.

We now state and proof our main theorem.

Theorem 2.5

Let C_i be a convex closed set in a complete metrisable locally convex space X for i = 1, 2, 3,…,n. Set $\bigcup_{i=1}^{n} C_i$. Let $f: C \to C$ be a map with $\overline{f(C)}$ compact and let $\varepsilon > 0$ be given. Then there exists a map $g: C \to C$ such that

(1) g has only a finite number of fixed points

(2) $F(f(x) - g(x)) < \varepsilon$ for all $x \in C$.

Proof

We use the technique of Baillon and Rallis [1]. We define the following open covering $\wp$ of C where $n_x < \varepsilon/2$ for all $x \in C$ and $S_r(x)$ denotes the open sphere centred on x with radius r.

$\wp = \{ S_{n_x}(x)$ where $x \in C_i \setminus \bigcup_{j \neq i} C_j$ and $S_{n_x}(x) \cap \bigcup_{j \neq i} C_j = \phi$ for $i = 1, 2,\ldots,n\}$

$\cup \{ S_{n_x}(x)$ where $x \in C_i \cap C_k \setminus \bigcup_{j \neq i,k} C_j$ and $S_{n_x}(x) \cap \bigcup_{j \neq i,k} C_j = \phi$ for i, k=1, 2, …,n, $i \neq k\}$

$\cup \{ S_{n_x}(x)$ where $x \in C_i \cap C_k \cap C_l \setminus \bigcup_{j \neq i,k,l} C_j$ and $S_{n_x}(x) \cap \bigcup_{j \neq i,k,l} C_j = \phi$ for i, k = 1, 2,..,n, $i \neq k$, $k \neq l$, $i \neq l\} \cup \ldots \cup$

$\{ S_{n_x}(x)$ where $x \in \bigcap_{i=1}^{n} C_i \}$.

This is a covering because for any $x \in C$, consider the set $\hat{C}_x = \bigcap_k \{C_k : x \notin C_k\}$. This set is closed and does not contain x. Then $C \setminus \hat{C}_x$ is open and so there exists $n_x > 0$ such that $S_{n_x}(x)$ is contained in $C \setminus \hat{C}_x$. The covering $\wp$ satisfies the following condition: (*) If $\bigcap_{k=l}^{l'} S_{n_{x_k}}(x_k) \cap C \neq \phi$, then the convex hull of $\{x_l \ldots x_{l'}\}$ is contained in C_m which is contained in C. To see this, note that since $\bigcup_{k=l}^{l'} S_{n_{x_k}}(x_k) \cap C \neq \phi$, there exists $z \in \bigcap_{k=l}^{l'} S_{n_{x_k}}(x_k) \cap C$ and $z \in C_m$ for some m, $1 \leq m \leq n$. Then by the definition of the covering $\wp$, each $x_j \in C_m$ for $j = l_1, l_2,\ldots,l'$. So the convex hull of $\{x_l..x_{l'}\}$ is contained in C. Since $\overline{f(C)}$ is compact, there exists a finite subcovering of $\overline{f(C)}$ by open sets relative to C, $\{ S_{n_{x_1}}(x_1) \cap C, S_{n_{x_2}}(x_2) \cap C, \ldots, S_{n_{x_k}}(x_k) \cap C \}$.

Let K denote the nerve of this covering, and |K| the geometric realization of K. By condition (*), $|K| \subseteq C$. We next construct the Schauder mapping. For $x \in C$, and i = 1, 2,…, .k, set $\mu_i(x) = \max(0, n_{x_i} - F(f(x)-x_i))$ and $\lambda_i(x) = \mu_i(x) \{ \sum_{i=1}^{k} \mu_i(x) \}^{-1}$. We define $\hat{f}$ for $x \in C$ by $\hat{f}(x) = \sum_{i=1}^{k} \lambda_i(x) x_i$. The function $\hat{f}$ is continuous and maps C into $|K|$. Moreover, $F(f(x) - \hat{f}(x)) < \varepsilon / 2$ for all $x \in C$. Since the finite polyhedron $|K|$ is also an ANR for $\varepsilon / 2$, there exists a $\delta > 0$ which satisfies the homotopy extension of Theorem 2.4. Further, there exists a $\beta > 0$ such that for any two maps $h, g: |K| \to |K|$ with $F(f-g) < \beta$, h and g are $\delta/2$-homotopic. By the Hop*f* construction, there exists a map $\hat{g}: |K| \to |K|$ such that $\hat{g}$ has only a finite number of fixed points and $F(\hat{f} \mid |K| - \hat{g}) < \beta$. So, $\hat{f} \mid |K|$ and $\hat{g}$ are $\delta/2$-homotopic. Let h_t denote the homotopy. We next define the homotopy $j_t: |K| \to |K|$ by

$$j_t = \begin{cases} h_{2t}, & 0 \le t \le 1/2 \\ h_{2-2t} & 1/2 \le t \le 1 \end{cases}$$

The map j_t is such that $j_o = \hat{f}\,||K|, j_1 = \hat{f}\,|\,|K|$ and $j_{1/2} = \hat{g}$. Further j_t is δ-homotopy. So, by Theorem 4, there exists an ε / 2-homotopy H_t: C → |K| such that $H_t||K| = j_t$ and $H_o = \hat{f}$ on C. Set $H_{1/2}$ = g. Then since $H_{1/2}||K| = \hat{g}$, g: C → |K| is a map with only a finite number of fixed points and F($\hat{f}$ (x) –g(x))<ε/2 for all x∈ C. Thus F(*f* - *g*)<ε.

C*orollary* 2.2 [1]

Let C_i be a convex closed set in a Banach space *X* for i = 1, 2, 3, …*n*. Set $\bigcup_{i=1}^{n} C_i$. Let f: *C* → *C* be a map with $\overline{f(C)}$ compact and let ε >0 be given. Then there exists a map *g*: *C* → *C* such that

(3) *g* has only a finite number of fixed points

(4) *F(f(x) –g(x))*< ε for all *x* ∈ *C.*

Remark

A multifunction *f*: C → C is a map from C to the set of all nonempty subsets of C. The range of f is f(C) = $\cup_{x \in C} f(x)$. An element x∈ C is said to be a fixed point of f if *x*∈ *f*(C) [13]. Is it possible for our Theorem to be generalized to a multifunction?

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