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Plasma heating by non-linear wave-Plasma interaction

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Abstract

We simulate the non-linear interaction of waves with magnetized tritium plasma with the aim of determining the parameter values that characterize the response of the plasma. The wave-plasma interaction has a non-conservative Hamiltonian description. The resulting system of Hamilton's equations is integrated numerically using fourth order Runge-Kutta scheme. It is found that for wave amplitude α as low as $0.01B_o$ the response of the plasma is remarkably different from the prediction of linear response theory. The response cannot be explained in terms of whether or not the wave frequency ω is a harmonic of the ion cyclotron frequency Ω. The scaled drift velocity of the ions $\tilde{\alpha}$ and the scaled phase velocity of the waves $\tilde{\beta}$ were found to be more relevant in explaining the response characteristics. For $\tilde{\alpha} >> \tilde{\beta}$, the plasma response is found to be chaotic while for $\tilde{\alpha} << \tilde{\beta}$, the response is either periodic or quasi-periodic. For $\tilde{\alpha} \approx \tilde{\beta}$ the waves do not interact with the plasma. The energy deposition (heating) by the waves in the plasma is found to be enhanced when the interaction occurs in the chaotic mode. In this mode, plasma diffusion is negligible suggesting that chaotic interaction of waves with plasma may enhance containment of the plasma.

Keywords: Wave-plasma-interaction, Phase-space, Poincare sections, Chaotic-response, Quasi-periodicity.

1.0 Introduction

Ion and electron cyclotron resonance (ICRH and ECRH) heating are a few of the many methods being studied for the heating of plasma to attain thermonuclear fusion temperatures. The ICRH and ECRH are essentially similar in concept, differing only in frequency range. The ICRH is however, by far the most important from the point of view of applications. The energy extracted by the electron component of the plasma is usually not available to the ion components. However, it is the ion-component that undergoes fusion. Using ECRH to heat plasma thus requires an extra mechanism that will modulate or couple the electron energy to ions. In spite of this shortcoming ECHR is being studied intensively for other reasons. Sources of waves in the ECRH regime are readily available from high-power microwaves from gyrotron and tunable free-electron lasers. Electron heating also finds direct use in providing thermal barriers in mirror devices as well as pressure profile modification in tokamaks [1].

In this paper we shall be interested in the direct heating of ions. Fast electrostatic waves in the ion-cyclotron resonance regime have been envisaged of being capable of heating ions [2]. The nature of the interaction is qualitatively different in the presence or absence of, and on the direction of propagation of the wave. In the absence or if the waves propagates in the direction of the magnetic field, heating mechanism is attributed to conventional Landau damping as a result of transient trapping in the potential trough of the waves. For small amplitude waves propagating across a magnetic filed, it is well known that interaction between the waves and ions occur only if the wave frequency ω is exactly some integral multiples of the ion cyclotron frequency Ω [3]. Under this condition, the phase space of ions consists of non-intersecting spiraling trajectories in contrast to closed orbits when there is no resonance. If the plasma is a multi-component one with different ion specie concentrations, heating may occur as a result of mode conversion from, magneto-sonic to electrostatic waves [1].

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The results of linear response theory are limited to very low amplitude waves. When the wave amplitude is appreciable, non-linearity enters the problem and one has to recourse to computer simulation to gain insight about the response of the system. In this paper we simulate the interaction of magnetized tritium plasma with finite amplitude waves in the ion-cyclotron frequency range. Previous attempts on this problem [4], [5], employed the use of some canonical transformation to suppress the explicit time dependency of the Hamiltonian of interaction thereby increasing the dimension of the problem with the dressed time coordinate τ. In other words the plasma and the waves have been combined into one conservative system. We think the response of the system is better demonstrated by letting the system (plasma) and open one i.e. in contact with a wave reservoir. This is more important from the point of view of application in that simulation can easily be used to characterize the parameters of the system and the reservoir that lead to specific response. This has become a widely used method even in the design of new materials. In section II we discuss briefly the theory of the model and the computational scheme, section III deals with the results and discussion while conclusion is in section 4.

2.0 Theory of the wave-plasma interaction model

We consider a tritium plasma placed in a uniform magnetic field, $B=B_o\vec{k}$, directed along the z-axis. Magnetic field in-homogeneity can introduce turbulence that may be difficult to differentiate from the non-linear nature of the wave plasma interaction [6], [7]. Electrostatic waves propagating in the x-direction, perpendicular to the magnetic field are applied to the plasma. Only low frequency waves of the ion-acoustic type are considered so that it is only the tritium ions that respond to the waves. The electrons formed a negative neutralizing background for the ions. The magnetic field and the electric field of the waves are furnished by the vector and scalar potentials A and Φ:

$$A = \frac{B_o}{2}(-y\vec{i} + x\vec{j}) \qquad (2.1)$$

$$\Phi = \varphi_o \sin(kx - \omega t) \quad , \qquad (2.2)$$

respectively, where B_o is the magnitude of the magnetic field, φ_o is amplitude of the wave electric potential while k and ω are the wave number and frequency of the wave. The $\nabla\times A$ and $-\nabla\Phi$ yield the magnetic field B and the electric field E of the waves respectively. By neglecting ion-ion interaction, the system is essentially a one ion system and may be described by the Hamiltonian, H

$$H = \frac{1}{2m}\left[(p_1 - \frac{\bar{Z}eB_o}{2}y)^2 + (p_2 + \frac{\bar{Z}eB_o}{2}x)^2 + p_3{}^2\right] + \bar{Z}e\varphi_o \sin(kx - \omega t) \qquad (2.3)$$

where, m is ion mass, p_1, p_2, p_3 are *x, y, z* components of the ion momentum, *e* is electron charge (for tritium ($\bar{Z}$ =1), *x, y* are positions , *t* is time and other symbols are as already defined.

The Hamilton canonical equations of motion are

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial P_1} = \frac{1}{m}(p_1 - \frac{eB_o}{2}y) \\
\dot{y} &= \frac{\partial H}{\partial p_2} = \frac{1}{m}(p_2 + \frac{eB_o}{2}x) \\
\dot{p}_1 &= -\frac{\partial H}{\partial x} = -\frac{1}{m}(p_2 + \tfrac{1}{2}eB_o x)\tfrac{1}{2}eB_o - \frac{ke\varphi_o}{m}\sin(kx - \omega t) \\
\dot{p}_2 &= \frac{\Omega}{2}(p_1 - \tfrac{1}{2}eB_o y)
\end{aligned} \qquad (2.4)$$

where $\Omega = \frac{eB_o}{m}$ is the ion cyclotron frequency and. P_3 is a constant and hence the *z* motion is a linear motion, which separates from the *x* and *y* motions, and it is of no further interest. Dimensionless variables were by measuring time in inverse Ω, length in inverse *k*, velocity in Ω/k so that equation (2.4) in dimensionless form becomes

$$\dot{X} = P_1 - \tfrac{1}{2}Y, \quad \dot{Y} = P_2 + \tfrac{1}{2}X$$
$$\dot{P}_1 = -\tfrac{1}{2}P_2 - \tfrac{1}{4}X - \alpha \sin(X - \nu\tau) \qquad (2.5)$$
$$\dot{P}_2 = \tfrac{1}{2}P_1 - \tfrac{1}{4}Y$$

where $\alpha = \dfrac{k\varphi_o/B_o}{\Omega/k} = \dfrac{E_o/B_o}{\Omega/k}$, $E_o=\varphi_o k$ is the amplitude of the electric field of the wave, $\tau=\omega t$ and $\nu = \dfrac{\omega}{\Omega}$. Equation (2.5) is the model of our wave plasma interaction. The equations are formally identical to the Lorentz force law equation if the Legendre type of transformation is used in expressing the Lagrangian in terms of the Hamiltonian *H*, i.e.

$$L = \sum_k \dot{q}_k p_k - H \qquad (2.6)$$

where in this case the mechanical momentum P_m is related to the canonical momentum P through

$$P_m = P + \frac{ZeB_o}{2}X \qquad (2.7)$$

Because the Hamiltonian is explicitly time dependent it does not fall into the general class of conservative non-linear system $\ddot{y} + g(y)\dot{y} + f(y)y = 0$ for which standard techniques such as operator splitting and integrals of motion are applicable [8]. The consequence is that error estimates are difficult to assess. One has to rely mostly on intuition and the hope that what works well for a reduced system may work well for the extended system.

Fourth order Runge-Kutta scheme was used in integrating equation (2.5). The phase space of the system is four dimensional (4-D), the display and interpretation of which is very difficult. Components of the phase space may be plotted but these often give wrong impression about the response of the system. Poincare method was used to project the 4-D phase space (X, Y, P_1, P_2) onto the 2-D phase plane (x, p_1). Starting from the surface Y = 0, with $P_2>0$, equation (2.5) was integrated numerically with initial conditions X_o, P_{1o} until $Y \approx 0$ with $P_2 > 0$ [9], [10]. The X_i and P_1 at this stage were extracted and plotted.

The response of the system was assessed qualitatively through visual examination of the phase space trajectories and their Poincare sections, quasi-linear diffusion theory and the guiding centre fluctuation. The rate of power dissipation by the waves in the plasma followed the quasi-linear approximation [3].

$$\frac{dP}{dt} = nm \int v^3 D \frac{\partial F}{\partial v} dv \qquad (2.8)$$

where *v* is the perpendicular velocity, *n* = number density, m = mass, and *F* is particle distribution function. The diffusion coefficient *D* was calculated using the formula

$$D = \frac{\pi E_o^{\,2}\omega^2}{2m^2k^2v^2} \sum_{i=-\infty}^{\infty} J_i(kv/\Omega)\delta(\omega - i\Omega) \qquad (2.9)$$

where J_i is the Bessel function of order *m* and δ is the Dirac delta function, other symbols are already defined. The asymptotic form of J_i of order 2 was used in equation (2.9) [11]. The equation above holds for resonance situation. When resonance is destroyed and turbulence takes over we computed the diffusion coefficient according to the formula.

$$D = \frac{e^2 Eo^{\,2}\omega^2}{2mk^2v^2} \frac{1}{(k^2v^2 - \omega^2)^{1/2}} \qquad (2.10)$$

To estimate the integral in equation (2.8), a random number generator generates an array of 1000 particle positions and momentum which were followed in time according to equation (2.5)). The number of oparticles lying in the velocity interval *v* and *v* + Δ*v* were counted and the integral was replaced by a sum over the particle velocities. An estimate of diffusion was also made using a particle guiding fluctuation

$$(\Delta r_{gc})^2 = (X(t) - \frac{P_1}{m\Omega})^2 + (Y(t) - \frac{P_2}{m\Omega})^2 - (X(0) - \frac{P_1}{m\Omega})^2 + (Y(o) - \frac{P_2}{m\Omega})^2 \qquad (2.11)$$

3.0 Results and discussion

The response of the plasma depends on the wave amplitude E_o, the strength of external magnetic field B_o and hence the cyclotron frequency Ω, the wave frequency ω, the initial conditions of the particles X_i, P_i and the wave number k. The electron charge e is assigned simulation value of 1C, $B_o = 1T$, tritium mass $m = 1g$, and the cyclotron frequency $\Omega/2\pi = 1Hz$; therefore $\alpha = E_o$ (magnitude). These choices, though arbitrary are reasonable because in application higher values of external magnetic field are rare and they are often fixed. Actual values like the electron charge $\sim 10^{-19}$C can easily cause floating point processing errors in computers. The tritium ion charge and mass are fixed constants, which cannot alter the dynamics except, by a scaling factor. The integration time step h =0.01sec was found to reproduce a closed elliptic orbit in the absence of waves which is in agreement with the analytic result of a charged particle motion in a uniform magnetic field. When the wave amplitude is infinitesimal but finite, $\alpha=0.01B_o$, the phase space trajectories for integral multiples of wave frequencies ω =2Ω, 3Ω, 4Ω,… remain elliptical as if there were no waves. However, for the same wave amplitude ($\alpha=0.01B_o$), the frequencies ω =0.5Ω, 0.8Ω, 2.5Ω (non-integral multiple of Ω) give rise to non-intersecting spiraling trajectories as shown in fig.1a (for ω =0.2Ω), meaning that for these frequencies the waves interact with the plasma. This result is contrary to the prediction of linear response theory that resonant interaction between waves and particles occurs only when the wave frequency is some integral multiple of the ion cyclotron frequency.

When the wave amplitude $\alpha = 0.5B_o$, and ω =1Ω, 2Ω, we observed that while ω =1Ω leads to intersecting trajectories, ω =2Ω does not even though both frequencies are integral multiples of Ω (fig. 1b and c). When $\alpha = 1B_o$, the phase space trajectories appear nested for ω =2.52347Ω and 2.3141Ω. In this case the wave frequency and the cyclotron frequency bears no common ratio. All these appear to have shown that frequency relationship alone in not sufficient to determine the response characteristics of the wave -plasma interaction.

Figures 1(a-c) are only (X,P_1) components of the 4-D phase space (X,Y,P_1,P_2). The other component (Y,P_2) have similar form. The components do not give a correct picture of the 4-D phase space. A projection of the $\Re^4 \rightarrow \Re^2$ by Poincare procedure are shown in figure 2(a-d). The trajectories that appeared complicated on the (X,P) plane showed quasi-periodicity for ω = 2.3141Ω and double quasi-periodicity for ω = 2.52347Ω (a and b respectively ; $\alpha = 1B_o$). However, for ω =0.5Ω, 2Ω ($\alpha=1B_o$), the Poincare sections appear chaotic as the set of points on the surface Y = 0 do not lie on any discernible closed curve (fig. 2c and 2d). Here one frequency is integral multiple of Ω while the other is not but both lead to chaotic response. Thus the Poincare sections too, have shown that the plasma response cannot be explained in terms of frequency relationship (ω,Ω) alone.

To account for the appropriate conditions for the response characteristics, we computed many Poincare sections corresponding to various ω, Ω and evaluated the empirical relation $\tilde{\beta} = \frac{1}{4}(\frac{\Omega}{\omega})^{1/3}\frac{\omega}{k}$. Table 1 gives a summary of the results plus appropriate remark judging from the visual observation of the Poincare plane (X,P_1).

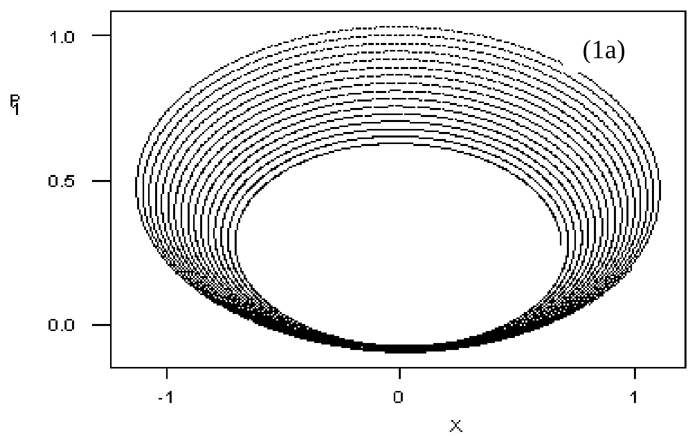
Table 1: Scaling values of ion drift and wave phase velocities.

ω/Ω	$\tilde{\alpha}$	$\tilde{\beta} = \frac{1}{4}(\frac{\Omega}{\omega})^{1/3}\frac{\omega}{k}$	Remark
0.2	0.01	0.0855	Q.P
0.5	0.01	0.1575	Q.P
0.5	0.1	0.1575	Q.P
0.5	1.0	0.1575	C
0.8	0.01	0.2154	Q.P
1.0	0.5	0.2500	S.C
1.0	2.0	0.2500	C
2.0	0.5	0.3969	Q.P
2.0	2.0	0.3969	C
2.5	0.5	0.4605	N.I
2.5	6.0	0.4605	C
2.52347	0.01	0.4638	Q.P

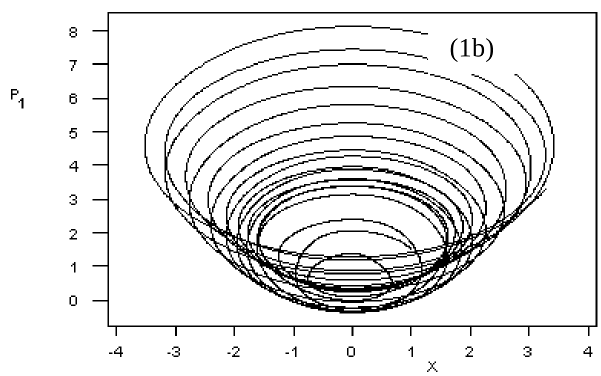
Q.P=quasi-periodic, C=chaotic, S.C=slightly chaotic, N.I=no interaction.

We deduce that for periodic or quasi-periodic orbits $\tilde{\alpha} << \tilde{\beta}$, where $\tilde{\alpha} = \frac{E_o / Bo}{\Omega / k}$ while for chaotic orbits

$\tilde{\alpha} >> \tilde{\beta}$. For $\tilde{\alpha} \approx \tilde{\beta}$ the waves do not interact with the plasma. From Equation (2.5) we see that $\tilde{\alpha}$ is the scaled drift velocity of the ions while $\tilde{\beta}$ as defined is the scaled phase velocity of the waves. In other words the wave plasma interaction is chaotic if the scaled ion drift velocity is much greater than the scaled wave phase velocity.



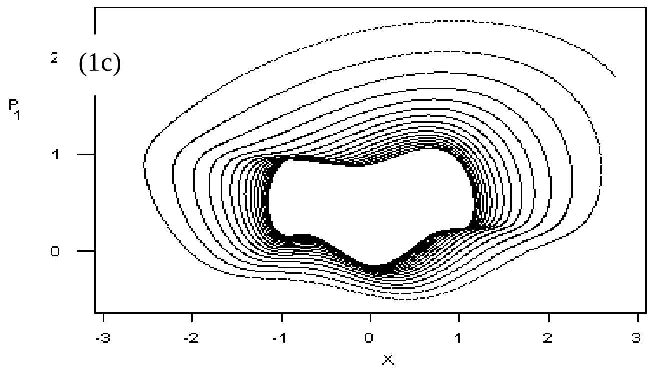
$\alpha = 0.01B_o$, $\omega = 0.5\Omega$



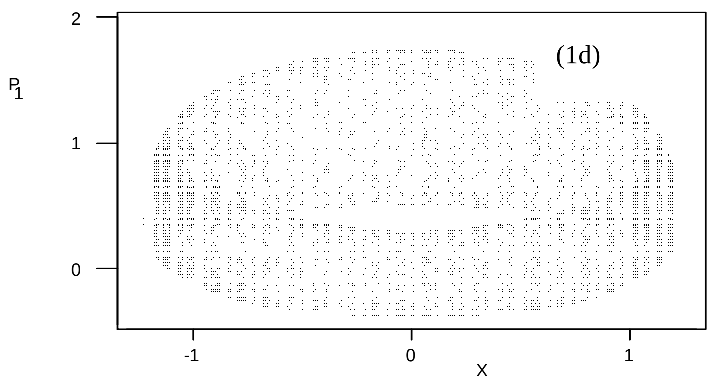
$\alpha = 0.5B_o$, $\omega = 1\Omega$

The energy dissipated by the waves in the plasma within the simulation time is numerically equal to the total area under the power-time curves. Figure 3(a and b) shows that the power-time curves for α =0.01B_o, ω=0.5, and α=2B_o, ω=2Ω respectively. The power dissipation in the plasma for $\tilde{\alpha} << \tilde{\beta}$ (for which the orbits are quasi-periodic, Figure.3a) is very small compared with that for which $\tilde{\alpha} >> \tilde{\beta}$ (chaotic orbits Figure (3b).

The guiding centre diffusion for α=0 (no incident waves) shows uniform oscillations about the guiding centre Figure 4a). For slightly chaotic interaction (α=0.5Bo, ω=1Ω), the oscillations damp out within half the simulation time (Figure.4b). For fully chaotic situation the guiding centre diffusion is nearly zero for more than three-quarter the simulation time. In the absence of external magnetic field the diffusion coefficient normally has linear time dependence [12]. The presence of magnetic field introduces oscillations in the guiding centre. The effect of linear time dependence of diffusion would be that of taking the plasma across the magnetic field to the containing vessel. Under chaotic interaction with the waves, the randominization of the phase space serves to mix the plasma thoroughly and the near zero diffusion may suggest enhanced containment of the plasma.

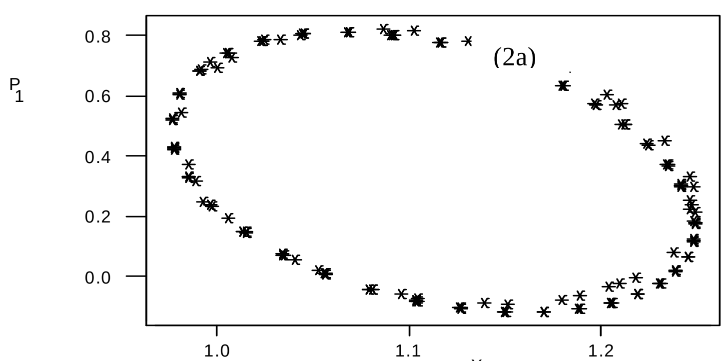


$\alpha = 0.5B_o$, $\omega = 2\Omega$

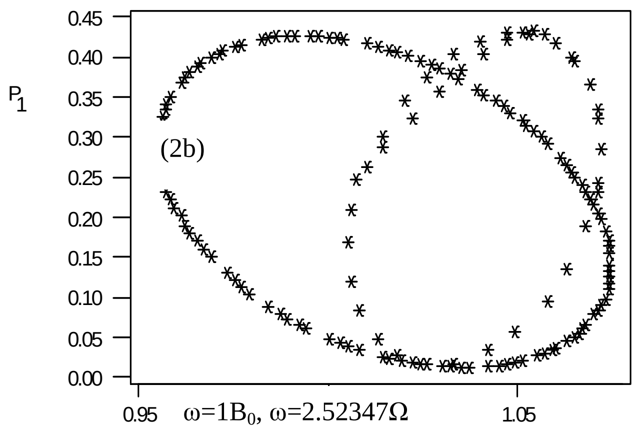


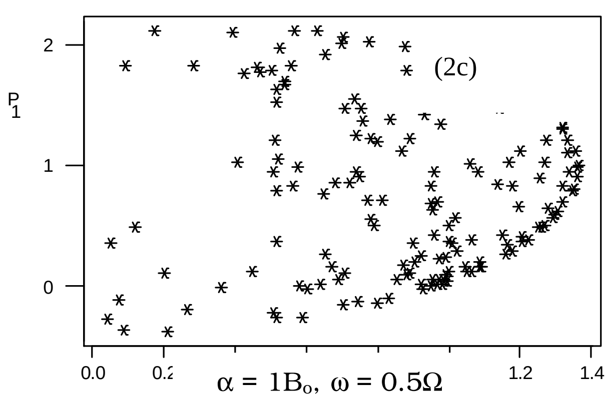
$\alpha = 1.0B_o$, $\omega = 2.3141$

Figure 1: *X-P*$_1$ component of the ion phase space



$\alpha = 1B_o$, $\omega = 2.3141\Omega$





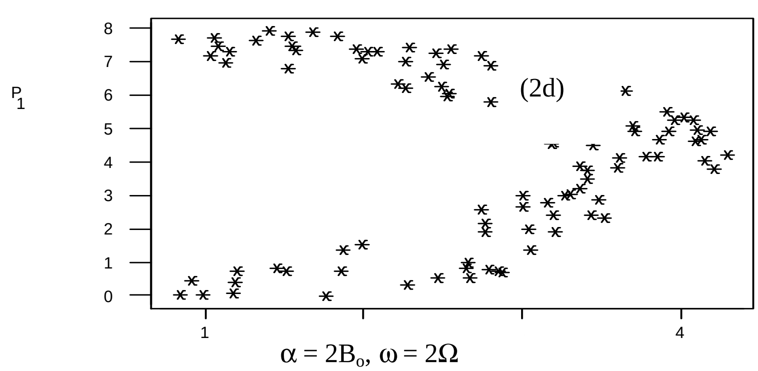
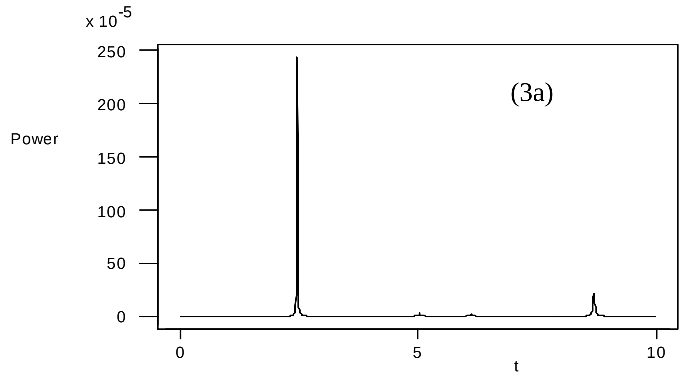


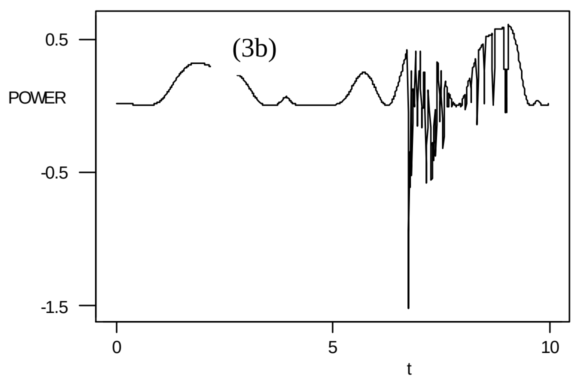
Figure 2: Poincare section of the 4-D phase space of ion

Note:

The initial conditions for the Poincare sections are $X = 1.0$, $Y = 0.0$, $P_1 = 0.2$, $P_2 = 0.5$



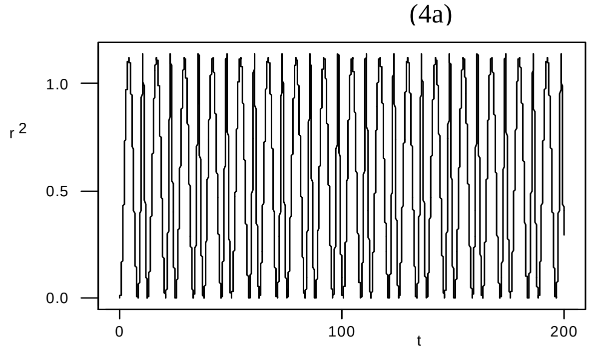
$\alpha = 0.01B_o,\ \omega = 0.5\Omega$



$\alpha = 2B_o,\ \omega = 2\Omega$

Figure 3: Power deposition profile.

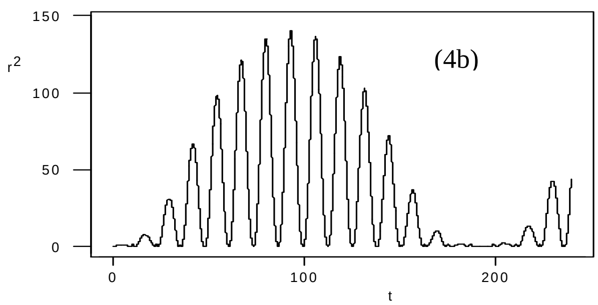
(4a)



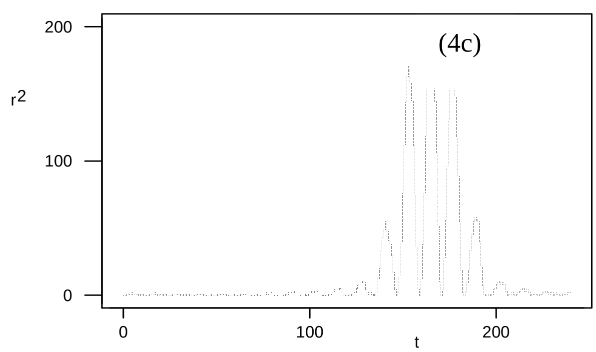
$\alpha = 0B_o$

4.0 Conclusion

We have found that wave-plasma interaction is a non-linear mechanism that cannot be explained in terms of the ion cyclotron frequency and the wave frequency alone. The interaction depends sensitively on the amplitude of the wave, the magnetic field strength as well as the phase velocity of the waves. For α as low as $0.01B_o$, the interaction has a remarkable departure from the presumptions of



$\omega = 0.5B_0$, $\omega = 1\Omega$



$\alpha = 2B_o$, $\omega = 2\Omega$

Figure 4: Guiding centre diffusion for wave-plasma interaction.

linear response theory. The nature of the interaction may be inferred from the scaled drift velocity, $\tilde{\alpha}$ of the plasma ions and the scaled phase velocity, $\tilde{\beta}$ of the waves. For $\tilde{\alpha} >> \tilde{\beta}$, the interaction is chaotic while for $\tilde{\alpha} << \tilde{\beta}$ the interaction is periodic or quasi-periodic. There is no interaction for $\tilde{\alpha} \approx \tilde{\beta}$. The heating of the plasma is substantially higher when the interaction occurs in the $\tilde{\alpha} >> \tilde{\beta}$ mode than in the $\tilde{\alpha} << \tilde{\beta}$. The near zero guiding centre diffusion under chaotic mode seems to suggest that chaotic wave-plasma interaction may enhance plasma containment.

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