Journal of the Nigerian Association of Mathematical Physics
Volume 10 (November 2006), **127 - 136**
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Measure synchronization in a coupled Hamiltonian associated with the motion of particles in a periodic potential

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Abstract

We report here, the existence of measure synchronization (MS) in a coupled Hamiltonian system associated with the motion of particles in a periodic potential of the pendulum type. We show that the oscillators can assume chaotic MS stares vis quasiperiodic measure desynchrononized state. In the chaotic MS state, the phase difference of the tow oscillators performs a stick-slip and random-walk-like motion analogous to the phenonomention of intermittency already established in the classical chaotic pendulum.

PACS: 05.45.Pq; 05.45. Xt; 05.45.Ac
Keywords*:* Measure synchronization; Hamiltonian systems; Chaos

1.0 Introduction

One of the fundamental non-linear phenomena observed in nature is synchronization. The intriguing concept of synchronization in non-linear systems is very relevant for a wide range of applications in Physics, Chemistry, Biology, secure communication and design of oscillator generators [1]. In the last decade, it has received much attention and some comprehensive reviews and books have appeared in the literature [1-4]. With the development of non-linear dynamics, the classical concept of synchronization has been extended from phase-looking of periodic oscillators to that of chaotic oscillators. The synchronization of chaotic systems, in particular presents a challenge since a chaotic system is extremely sensitive to small perturbations in initial conditions.

In contemporary literature, many kinds of chaos synchronization have been well described. These includes complete synchronization (CS) [5-8], lag synchronization (LG) [9], phase synchronization [9-13], anticipated synchronization (AS) [14, 15] and measure synchronization (MS) [16-21]. Here, we have only referred to few relevant literatures in these directions. So far, most of the studies in this field have focused on dissipative systems. However, in a recent study, Hampton and Zanette [16] presented the concept of measure synchronization (MS) between identical Hamiltonian systems. Since the, some researches have investigated the phenomenon of MS in coupled Hamiltonian systems [17-21]. The main characteristics of MS are that two oscillators share the same phase space with the same identical invariant measure, though they are not strictly synchronized in the original sense of synchronization.

Hamiltonian systems are very significant because many practical systems can be well approximated by Hamiltonian formalism even at weak dissipation. Since there is a direct connection between any classical Hamiltonian system and its quantum version, it is possible to extend the concept of MS and the controlling approach to quantum system [19]. Thus, investigation the behaviour of MS in coupled Hamiltonian systems is beneficial in the understanding of its possible link with quantum systems.

In previous studies, we show the existence of MS and partial MS phenomenon, in a Hamiltonian system associated with the Nonlinear Schrödinger Equation [20, 22] and the Duffing Hamiltonian system [21] respectively. While the study of synchronization in coupled Hamiltonian systems has remained an open research field that has received relatively less attention, in this present paper, we examine this issue in the context of a familiar Hamiltonian system associated with the motion of a particle in a periodic potential of the pendulum type [23].

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2.0 Model and formulation

Let the Hamiltonian describing the motion of a particle in a periodic potential be given by:

$$H = \frac{p_i^2}{2m} + V(q_i) + K\frac{(q_{i+1} - q_i)^2}{4} \quad (2.1)$$

Here,

$$V(q_i) = 1 - \cos q_i \quad (2.2)$$

is the periodic potential associated with the oscillation of the pendulum [23]. q_i is the generalized coordinates, m is the mass taken to be unity and p_i is the generalized momentum. The last term (coupling term) on the right hand side of equation. (2.1) is equivalent to the interaction energy, with K being the coupling parameter that determines the strength of the coupling.

Substituting equation (2.2) in equation (2.1), the resulting Hamilton's equations of motion take the form:

$$\dot{q}_i = p_i, \quad p_i = \sin q_i + K(q_{i+1} + q_{i-1} - 2q_i) \quad (2.3)$$

Let us consider a simple case of two oscillators wherein $i = 1,2$. For $i = 1$ and considering the allowed regions, then $q_0 = q_1$, equation (2.3) reads:

$$\dot{q}_i = p_i, \quad p_i = \sin q_i + K(q_2 - q_1). \quad (2.4)$$

Similarly, when $i = 2$, we have:

$$\dot{q}_i = p_2, \quad p_1 = \sin q_2 + K(q_1 - q_2). \quad (2.5)$$

Equations (2.4) and (2.5) are a set of coupled canonical equations derived from the non-integrable non-linear Hamiltonian (1) and therefore cannot be solved analytically.

Thus, we employed the standard Fourth order Runge-Kutta routine to solve equations (2.4) and (2.5) numerically. The dynamics of systems (2.4) and (2.5) depends on K as well as on its initial conditions. By varying the coupling strength, K, the total energy can be regarded as an irrelevant parameter by suitable scaling. Thus, we fix the total energy $H = 2.5$ throughout the paper. Also, we fix $q_i(t = 0) = 0$, a configuration which ensures that the interaction energy

$$E_i = K(q_{i+1} + q_{i-1} - 2q_i) \quad (2.6)$$

is given zero initial value at $K = 0$, so that any slight adjustment of K does not change the total energy, for any initial choices of $p_i(0)$. Thus, in our model, there are two adjustable parameter- the initial conditions p_i and the coupling strength K.

3.0 Result and discussion

We simulated the coupled systems (4) and (5) and studied its dynamical behaviour using the initial conditions and the coupling parameter K as the control parameters. We found that three kinds of MS states: periodic, chaotic and quasiperiodic can be observed depending on initial conditions. Here, we report on the chaotic MS states intermingled with quasiperiodic MS states. The initial conditions for the coupled systems were set as follows: $q_1(0) = q_2(0) = 0, p_1(0) = 0.1$ and $p_2(0) = \sqrt{2E - p_1^2}$. This configuration ensures that the initial trajectories assume a double-well orbit so that a cross-well chaos can be obtained as employed in ref. [21]. We note that other configurations can lead to single-well trajectories. For instance, by setting $q_1(0) = 0.5$ and retaining the other initial conditions, one of the oscillators would be confined to the positive potential well at $q_{1,2} = 2.0$.

We start by plotting in Fig. 2(a) the periodic orbits for oscillators (1) and (2) for zero coupling, that is $K = 0$. When a small non-zero coupling is switched on, we found that the external layer of the oscillator (1) approaches the external layer of oscillator (2) (and vice versa) at the initial boundary as shown in Fig. 2 (b); here $k = 0.0025$. The oscillators are quasiperiodic and no MS achieved. The time series of the q_1, shown in Fig. 3(a) and (b) respectively confirms the periodicity and quasiperiodicity of the orbits in Fig. 2(a) and (b). In the case where $q_1(0) = 0.5$ (not shown), the interaction via the coupling enables the oscillator (1) undergoes a tunneling across the potential hill at $q_i = 0$. When the coupling strength K is gradually increased to 0.1, the two orbits begin to share the same phase space due to coupling in their chaotic state.

As the coupling strength is further increased above $K = 0.1$, different topological structures which are essentially quasiperiodic and chaotic MS states can be captured. For instance, for $K = 0.5$, we plot the trajectories corresponding to the two orbits in their chaotic MS states. In Fig. 3 (c), we justify the chaotic behaviour by showing the irregular time evolution of the generalized coordinate, q_1 for $K = 0.5$.

In order to clarify the MS transitions, we calculated the bare energies $h_i = \langle E_i \rangle, i = 1,2$

$$h_{1,2} = \frac{1}{T}\int_0^T E_{1,2}(t)dt \qquad (3.1)$$

$$E_{1,2}(t) = \frac{p_{1,2}^2}{2} + \sin q_{1,2}, \qquad (3.2)$$

and the average interaction energy

$$h_{1,2} = \frac{1}{T}\int_0^T K\,[q_1(t) - q_2(t)]^2\,dt \qquad (3.3)$$

In Figure 4(a), it can be seen that there is finite difference between h_1 and h_2 below the transition critical coupling $K_c = 0.1$ while above K_c, both oscillators begins to assume relatively identical bare energy. To reach $h_1 = h_2$ in the chaotic state, extremely long time run is required, this explains why $h_{1,2}$ fluctuate just after the MS transition in Fig. 4(a). Although Fig. 4(b) appears to depict no relationship between K and average interaction energy h_1, it can be observed that h_1 increases before the MS transition regime and decreases monotonically after the transition. Wang et al [17] have conjectured that such behaviour can be explained based on the average phase difference between the two oscillators defined by

$$\begin{aligned} \langle|\Delta\theta|\rangle &= \frac{1}{T}\int_0^T |\,\Delta\theta(t)\ m|dt, \\ \Delta\theta(t)_m\ \ & sgn\ [\Delta\theta(t)]\,[\pi - |\Delta\theta(t) \bmod \pi|\,] \\ \Delta\theta(t) &= \theta_1(t) - \theta_2(t), \\ \theta_i(t) &= \arctan\ (p_i / q_i) \in [0, 2\pi], \ i = 1,2. \end{aligned} \qquad (3.4)$$

In equation (3.4), $\theta_{1,2}(t)$ is defined in the range $[0,2\pi]$ and $\Delta\theta(t)_m$ is define in the range $[-\pi,\pi]$ son as to indicate the relative positions of two oscillators in the phase plane. In Figure 4(c), we show that the phase difference $\langle|\Delta\theta|\rangle$ increases monotonically before $K_c = 0.1$, until it reaches a peak value (≈ 1.35) at K_c. Beyond K_c, $\langle|\Delta\theta|\rangle$ also decreases monotonically with intermittent discontinuities, i. e. Stick-slip and random-walk-like motion. The discontinuities are analogous to intermittent synchronization associated with de-synchronous activities in dissipative systems [24-27]. Thus, implying that the oscillators can never phase-lock in the chaotic MS state.

4.0 Concluding Remarks

In summary, we have shown the existence of measure synchronization (MS) in a Hamiltonian system associated with the motion of particles in a periodic potential of the pendulum type; particular emphasis on the transition to chaotic MS state. The classical equation of motion of the system considered here has been widely studied in the field of nonlinear dynamics and the phenomenon of intermittent synchronization have been validated theoretically, numerically and experimentally [24-26]. Our numerical findings for the Hamiltonian counterpart reveal that three probable MS states can be reached by this system, namely periodic, quasiperiodic and chaotic MS states. The transition to chaotic MS State via quasiperiodicity (QP) has been characterized using measurable statistical quantities. While long time run is practically required for the bare energies of the system to be identical on the one hand, the oscillators could not permanently phase-lock; rather perform a stick-slip and random-walk-like motion. Thus, confirming the phenomenon of intermittency in the chaotic pendulum.

Acknowledgement

Dr. U. E. Vincent is grateful to Dr. Anatole Kenfack and the Max Planck Institute for the Physics of Complex System, Dresden Germany, for updating and supplying us with current literature in this domain.

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Figure 3: Time evolution of the generalized coordinate, q_1, for (a) periodic regime with no coupling, $K = 0$ (b) Quasiperiodic regime, $K = 0.0025$ (c) Chaotic regime with strong coupling, $K = 0.5$.

Figure 4: (a) Average bare energies of equations. (3.1) and (3.2), $h_{1,2}$ vs K. MS sets in at $K = 0.1$ where discontinuous jumps of $h_{1,2}$ can be identified. (b) Average interaction energy h_I of equation (3.3) plotted against K. (c) Phase difference $\langle|\Delta\theta|\rangle$ vs K showing discontinuities.

Figure 1: The Periodic Potential

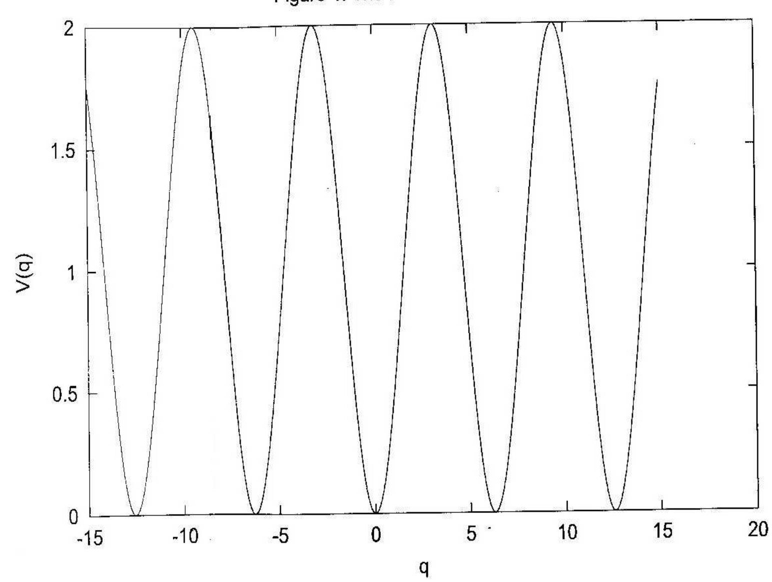


Figure 2(a)

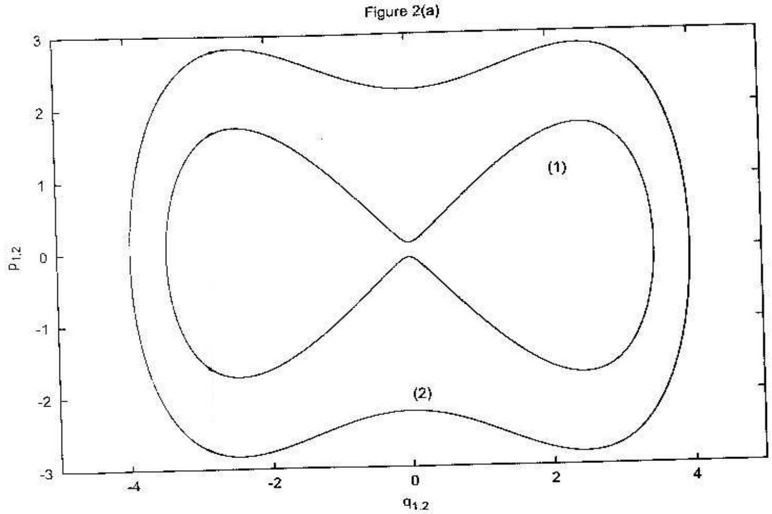
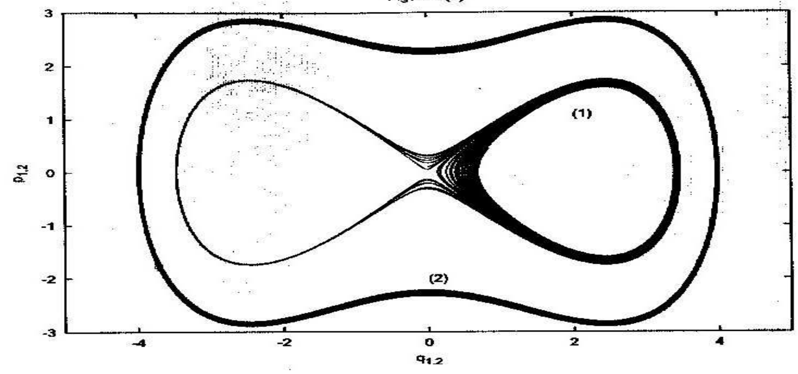


Figure 2(b)



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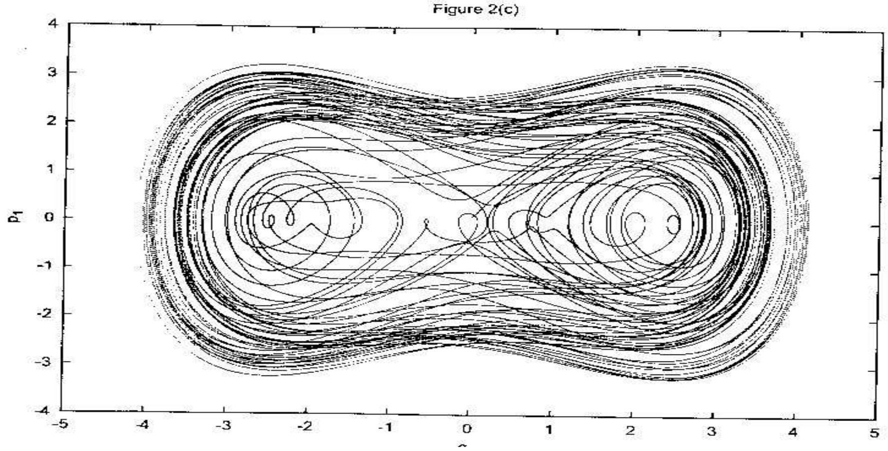

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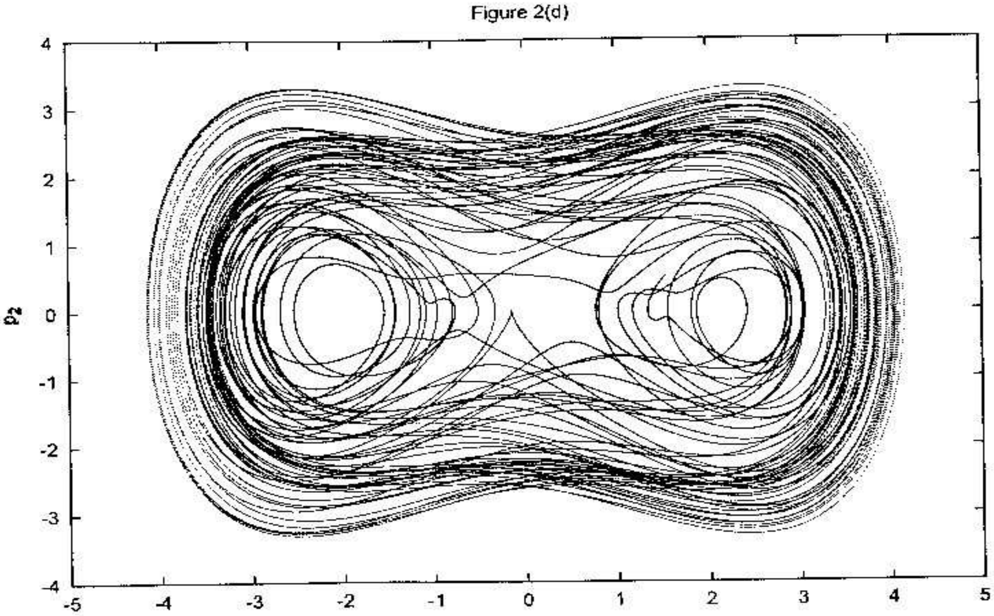
Figure 2(d)



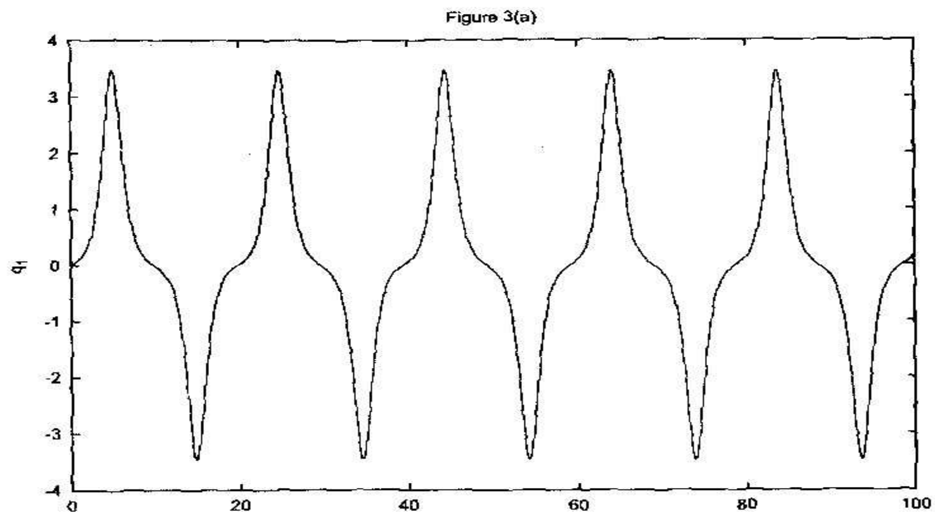


Figure 3(a)
4
3
2
1
0
-1
-2
-3
-4
q_1
0
20
40
60
80
100

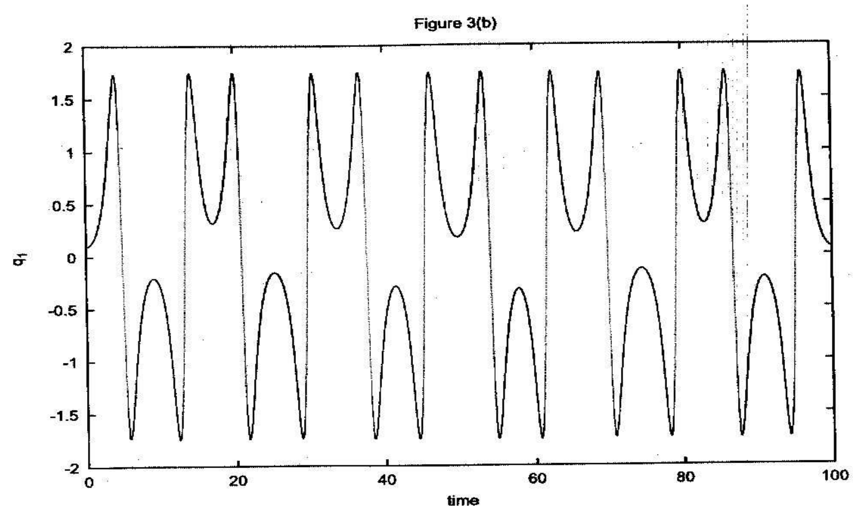


Figure 3(b)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
q_1
0
20
40
60
80
100
time

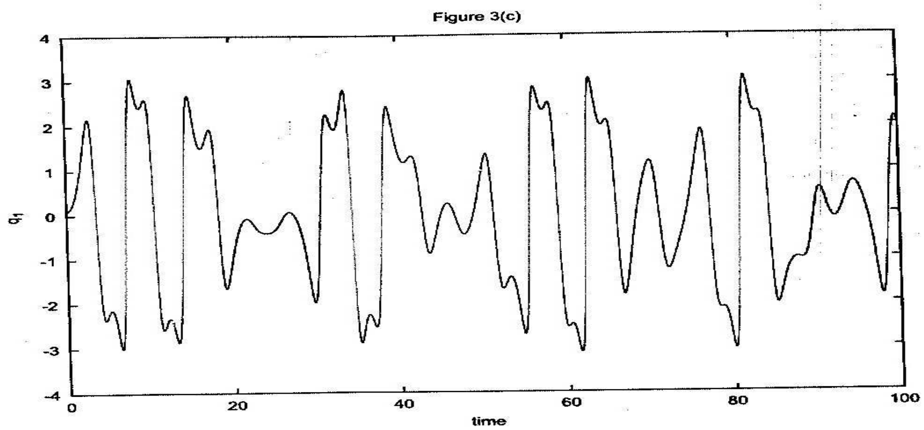


Figure 3(c)
q1
time

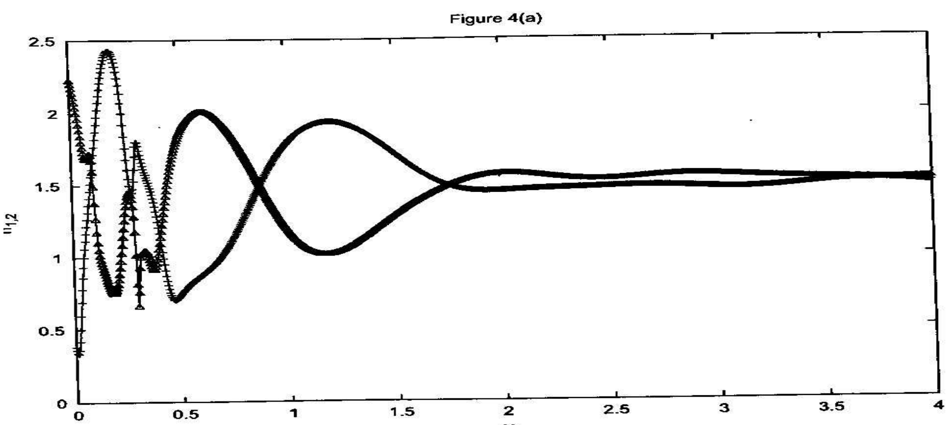


Figure 4(a)

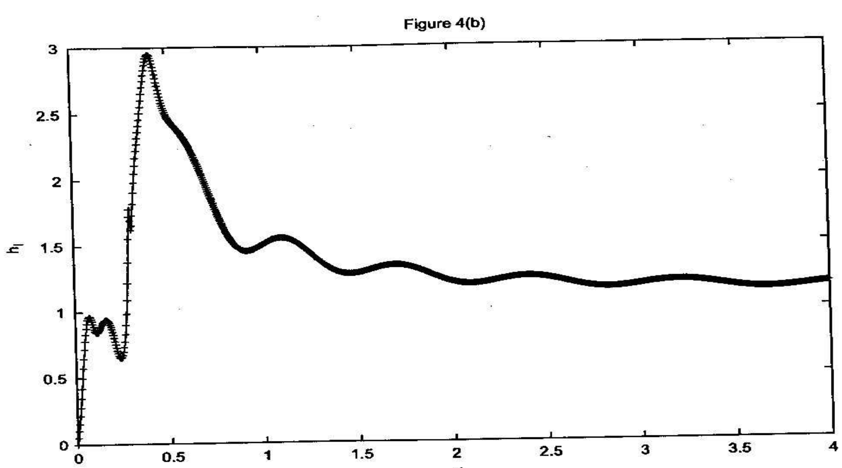
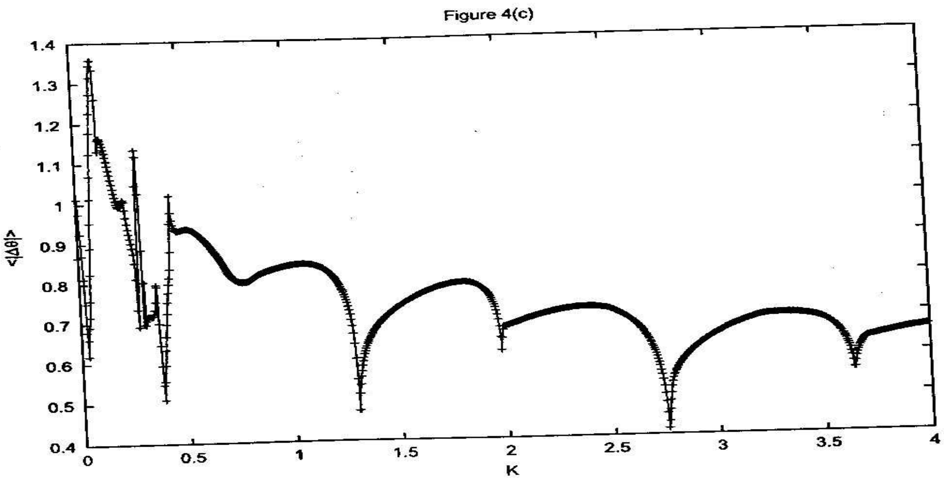


Figure 4(b)
K



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