Journal of the Nigerian Association of Mathematical Physics
***Volume 10 (November 2006),* 91 -96**
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Variable order one-step methods for initial value problems I

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Abstract

A class of variable order one-step integrators is proposed for Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs). It is based on a rational interpolant.

1.0 Introduction

Let us consider the initial value problem

$$y' = f(x, y),\ y(a) = y_0 \in R^m \ ,\ x \in [a, b],\ y(x) \in R^m \qquad (1.1)$$

whose solution may be singular

Niekerk (1987) extended the idea of Fatunla (1982) by using the interpolant

$$y(x) \approx B + \frac{A}{1 + b_1 x} \qquad (1.2)$$

to derive a one-step method which can be applied to the initial value problem (1.1) without any restrictions on the initial value. Then, in an attempt to generalize (1.2) he first considered the interpolant

$$y(x) \approx B + \frac{A}{1 + b_1 x + b_2 x^2} \qquad (1.3)$$

But the difficulties he encountered for solving for the coefficients $A,\ B, b_1\ and\ b_2$ made him to abandon the idea of the generalization.

To resolve the difficulties imposed by the nature of the interpolant proposed by Nickerk (1987), Ikhile (2001) considered the interpolant

$$y(x) \approx B + \frac{Ax}{1 + \sum_{j=1}^{k} b_j x^j}\ ,\ k \geq 1 \qquad (1.4)$$

and derived a family of schemes of order $k + 1$

It is the purpose of the present paper to show how more accurate solutions may be obtained using the interpolant.

$$y(x) \approx A + \frac{a_0 x + a_1 x^2}{1 + \sum_{j=1}^{k} b_j x^j}\ ,\ k \geq 1 \qquad (1.5)$$

The proposed scheme which is an extension of the concept of the development of Ikhile (2001) is a class of variable order one-step rational methods. While on the other hand the scheme derived by Otunta and Nwachkwu (2005) is just a single rational one-step method of order 6.

2.0 Derivation of the integrator

We interpolate the theoretical solution of (1.1) by

$$y(x) \approx A + \frac{a_0 x + a_1 x^2}{1 + \sum_{j=1}^{k} b_j x^j}, \; k \geq 1 \qquad (2.1)$$

The resultant variable-order one-step scheme is given by

$$y_{n+1} = A + \frac{a_0 x_{n+1} + a_1 x_{n+1}^2}{1 + \sum_{j=1}^{k} b_j x_{n+1}^j}, \; k \geq 1 \qquad (2.2)$$

The parameters $A, a_0, a_1 \, and \, b = (b_1, b_2, \cdots . b_k)^T$ are determined by recasting (2.2) in the form

$$y_{n+1} = A + \left(a_0 x_{x+1} + a_1 x_{n+1}^2\right)\left[1 + \sum_{r=1}^{\infty} (-1)^r \left(\sum_{j=1}^{k} b_j x_{n+1}^j\right)^r\right] \qquad (2.3)$$

(Using the binomial series)
Adopting the idea of Otunta and Ikhile (1996, 1999), (2.3) becomes

$$y_{n+1} = A + \left(a_0 x_{n+1} + a_1 x_{n+1}^2\right)\left[1 + \sum_{r=1}^{\infty}\left(\sum_{j=r}^{k \cdot r} C_j^{(r-1)} x_{n+1}^j\right)(-1)^r\right] \qquad (2.4)$$

where

$$\left(\sum_{j=1}^{k} b_j x^j\right)^r = \sum_{j=r}^{k \cdot r} C_j^{(r-1)} x^j \qquad (2.5)$$

$$C_s^{(q)} = \sum_{j=1}^{s} C_{s-j}^{(q-1)} b_j, \; s = (q+1)(1)(q+1)k, \; C_j^{(0)} = b_j$$

Then comparing (2.4) termwise in powers of h with

$$y(x_{n+1}) = \sum_{j=0}^{\infty} \frac{h^j y_n^{(j)}}{j!}, \; y_n^{(0)} = y_n \qquad (2.6)$$

we obtain the order equations:

$$A = y_n$$

$$x_{n+1} a_0 = h y_n^I$$

$$x_{n+1}^2 (a_1 - a_0 b_1) = \frac{h^2 y_n^{II}}{2!}$$

$$x_{n+1}^3 \left(- a_0 b_2 - a_1 b_1 + a_0 c_2^{(I)}\right) = \frac{h^3 y_n^{III}}{3!}$$

$$x_{n+1}^4 \left(- a_0 b_3 - a_1 b_2 + a_0 c_3^{(1)} + a_1 c_2^{(1)} - a_0 c_3^{(2)}\right) = \frac{h^4 y_n^{(4)}}{4!} \qquad (2.7)$$

$$x_{n+1}^5 \left(- a_0 b_4 - a_1 b_3 + a_0 c_4^{(1)} + a_1 c_3^{(1)} - a_0 c_4^{(2)} - a_1 c_3^{(2)} + a_0 c_4^{(3)}\right) = \frac{h^5 y_n^{(5)}}{5!}$$

$$x_{n+1}^6 \left(- a_0 b_5 - a_1 b_4 + a_0 c_5^{(1)} + a_1 c_4^{(1)} - a_0 c_5^{(2)} - a_1 c_4^{(2)} + a_0 c_5^{(3)} + a_1 c_4^{(3)} - a_0 c_5^{(4)}\right) = \frac{h^6 y_n^{(6)}}{6!}$$

$$x_{n+1}^7 \left(- a_0 b_6 - a_1 b_5 + a_0 c_6^{(1)} + a_1 c_5^{(1)} - a_0 c_6^{(2)} - a_1 c_5^{(2)} + a_0 c_6^{(3)} + a_1 c_5^{(3)} - a_0 c_6^{(4)} - a_1 c_5^{(4)} + a_0 c_6^{(5)}\right) = \frac{h^7 y_n^{(7)}}{7!}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$x_{n+1}^k \Bigg(a_{k-1} - \sum_{j=0}^{k-1} a_j b_{k-1-j} + \sum_{j=0}^{k-2} a_j c_{k-1-j}^{(1)} - \sum_{j=0}^{k-3} a_j c_{k-1-j}^{(2)} + \sum_{j=0}^{k-4} a_j c_{k-1-j}^{(3)} - \sum_{j=0}^{k-5} a_j c_{k-1-j}^{(4)}$$

$$+ \ldots + (-1)^{k-2} \sum_{j=0}^{1} a_j c_{k-1-j}^{(k-3)} + (-1)^{k-1} a_0 c_{k-1}^{(k-2)} \Bigg) = \frac{h^k y_n^{(k)}}{k!}$$

$$x_{n+1}^{k+1}\left(a_k-\sum_{j=0}^{k} a_j b_{k-j}+\sum_{j=0}^{k-1} a_j c_{k-j}^{(1)}-\sum_{j=0}^{k-2} a_j c_{k-j}^{(2)}+\sum_{j=0}^{k-3} a_j c_{k-j}^{(3)}-\sum_{j=0}^{k-4} a_j c_{k-j}^{(4)}+\ldots+(-1)^{k-1}\sum_{j=0}^{1} a_j c_{k-j}^{k-2}+(-1)^k a_0 c_k^{k-1}\right)$$
$$=\frac{h^{k+1}y_n^{(k+1)}}{(k+1)!}$$

$$x_{n+1}^{(k+2)}\left(a_{k+1}-\sum_{j=0}^{k+1} a_j b_{k-j+1}+\sum_{j=0}^{k} a_j c_{k-j+1}^{(1)}-\sum_{j=0}^{k-1} a_j c_{k-j+1}^{(2)}+\sum_{j=0}^{k-2} a_j c_{k-j+1}^{(3)}-\sum_{j=0}^{k-3} a_j c_{k-j+1}^{(4)}+\ldots+(-1)^k\sum_{j=0}^{1} a_j c_{k-j+1}^{k-1}+(-1)^{k+1} a_0 c_{k+1}^{(k)}\right)$$
$$=\frac{h^{k+2}y_n^{(k+2)}}{(k+2)!}$$

with $a_k = 0 \quad for \quad k \geq 2$

The associated local truncation error $Lte_n = y_{n+1} - y(x_{n+1})$ is given by

$$Lte_n = x_{n+1}^{(k+3)}\left(-\sum_{j=0}^{k+2} a_j b_{k-j+2}+\sum_{j=0}^{k+1} a_j c_{k-j+2}^{(1)}-\sum_{j=0}^{k} a_j c_{k-j+2}^{(2)}+\sum_{j=0}^{k-1} a_j c_{k-j+2}^{(3)} -\sum_{j=0}^{k-2} a_j c_{k-j+2}^{(4)}+\ldots+(-1)^{k+1}\sum_{j=0}^{1} a_j c_{k-j+2}^{k}+(-1)^{k+2} a_0 c_{k+2}^{(k+1)}\right)-\frac{h^{k+3}y_n^{(k+3)}}{(k+3)!} \tag{2.8}$$

where y_{n+1} *and* $y(x_{n+1})$ are the computed and theoretical solution of (1.1) respectively and $x_n = nh$ with the solution $y(x)$ having derivatives to the desired order and h is the step size. The attainable order of the scheme is at least $p = k + 2$. From (2.7) the parameters of the rational scheme (2.2) are given by $A = y_n$ $\quad a_0 = \frac{h\, y_n^I}{x_{n+1}}$

(2.9)

$a_1 = \frac{h^2 y_n^{II}}{2!\, x_{n+1}^2} + \frac{h\, y_n^I}{x_{n+1}} b_1$. We obtain $b = (b_1, b_2, \ldots, b_k)^T$ from the system of quations $Ab = -C$, where

$$A = \lfloor A_{ij} \rfloor\; i, j = 1(1)k \qquad C = (c_1, c_2, \ldots c_k)^T \tag{2.10}$$

$$b = (b_1, b_2, \ldots b_k)^T \text{ with } A_{ij} = \begin{cases} 0, & k-j+1 \leq 0 \\ \dfrac{h^{k-j+1}\, y_n^{(k-j+1)}}{(k-j+1)!\, x_{n+1}^{(k-j+1)}}, & otherwise \end{cases}, \quad C = \frac{h^{k+1}\, y_n^{(k+1)}}{(k+1)!\, x_{n+1}^{k+1}}$$

The derivation of these parameters are devoid of the solution of the linear systems of equations unlike Fatunla (1982) and Luke et al (1975). They are component applicable to systems of ODEs of (1.1) in the sense of Lambert (1974)

3.0 Stability function

The stability function $\mu(z)$ for (2.2), using

$$y_{n+1} = \mu(z) y_n \tag{3.1}$$

For $k = 1$
$$\mu(z) = \frac{6 + 4z + z^2}{6 - 2z} \tag{3.2}$$

For $k = 2$
$$\mu(z) = \frac{12 + 6z + z^2}{12 - 6z + z^2} \tag{3.3}$$

For $k = 4$
$$\mu(z) = \frac{720 + 360z + 60z^2 - z^4}{720 - 360z + 60z^2 - z^4} \tag{3.4}$$

The functions are obtained by applying (2.2) on the test problem $y' = \lambda y$, $\quad R_e(\lambda) < 0$

The methods are A- stable.

4.0 Numerical Experiment

Numerical results on some problems are presented

Problem (1)

Fatunla (1986), Niekerk (1987), Lambert (1974), Luke et al (1975), Ikhile (2001),

$$y^1 = 1 + y^2 \ , \ y(0) = 1, \ 0 \le x \le 1$$

$$y(x) = \tan\left(x + \frac{\pi}{4}\right) \qquad (4.1)$$

Table 1. Performance of some algorithms on the IVP (4.1); $h = 0.05$, $0 \le x \le 1$, **/error/**

x	**Best of Fatunla (1986)**	**Best of Van Niekerk (1987)**	**Best of Lambert & Shaw (1965)**	**Best of Luke et al (1975)**
0.10	$P = 2$ 8.277 (-4)	$P = 2$ 2.0 (-6)	8 (-8)	2 (-4)
0.20	$P = 4$ 5.930 (-5)	$P = 2$ 8.0 (-7)	8 (-7)	5 (-4)
0.30	$P = 5$ 2.303 (-5)	$P = 2$ 2. 0 (-5)	4 (-7)	1 (-3)
0.40	$P = 5$ 2. 408 (-5)	$P = 2$ 5.0 (-5)	7 (-7)	2 (-3)
0.50	$P = 9$ 1.337 (-6)	$P = 2$ 1.0 (-4)	2 (-6)	5 (-3)
0.60	$P = 9$ 4.552 (-6)	$P = 2$ 5.0 (-4)	4 (-6)	1 (-2)
0.65	$P = 9$ 1.04 (-5)	$P = 2$ 1. 0 (-3)	8 (-6)	2 (-2)
0.70	$P = 9$ 7.349 (-5)	$P = 2$ 3. 0 (3)	2 (-5)	8 (-2)
0.75	$P = 9$ 1.017 (-5)	$P = 2$ 2.0 (-2)	2 (-2)	4 (-1)
0.80	$P = 9$ 1.585 (-2)			
0.90	$P = 3$ 6.093 (-3)			
1.00	$P = 3$ 9. 291 (-3)			

x	**Best of Otunta & Ikhile (1999)**	**Best of Ikhile (2001)**	**Best of Schemes (2.2)**
0.10	2.420 (-7)	$k = 2$ 3.5 (-8)	$k = 3$ 2.8535150218359080 (-8)
0.20	2.893 (-7)	$k = 2$ 9. 1 (-8)	$k = 4$ 4.747200326662025 (-8)
0.30	6. 972 (-7)	$k = 2$ 1.9 (-7)	$k = 4$ 5.298598494966551 (-8)
0.40	1. 601 (-6)	$k = 2$ 3.93 (-7)	$k = 4$ 6.277796725336237 (-8)
0.50	3. 970 (-6)	$k = 2$ 8.76 (-7)	$k = 4$ 8.096290013321106 (-8)
0.60	1. 562 (-5)	$k =2$ 2.541 (-6)	$k = 4$ 1.2073994124121360 (-7)
0.65	2.803 (-5)	$k = 2$ 4.952 (-6)	$k = 4$ 1.6356842612788940 (-7)
0.70	6.866 (-5)	$k= 2$ 1.3356 (-5)	$k = 4$ 2.5744784013050410 (-7)
0.75	4.867 (-4)	$k = 2$ 8.3117 (-5)	$k = 4$ 6.1864210070354990 (-7)

0.80	2.828 (-3)	$k = 2$ 5.2088 (-4)	$k = 4$ 1.49882063576801400 (-6)
0.90	5.382 (-5)	$k = 2$ 9. 554 (-6)	$k = 3$ 6.3593217293414607 (-8)
1.00	1.807 (-5)	$k= 2$ 3.061 (-6)	$k = 3$ 3.1993103564840417 (-8)

Our schemes (2.2) exhibit remarkable improvement over most existing methods.

Problem (2):

Tam (1989), DETEST [HULL et al (1972)]

$$y^1 = \frac{-y^3}{2}, \; y(0) = 1, \; y(x) = \frac{1}{\sqrt{1+x}} \qquad (4.2)$$

Table 2: Performance of some algorithms on the IVP (4.2);/error/

Best of Ikhile (2001)			Best of Schemes (2.2)	
x	$h = 0 \cdot 1$	$h = 0 \cdot 01$	$h = 0 \cdot 1$	$h = 0 \cdot 01$
1.	$k = 2$ 3.761987 (-5)	$k = 2,3$ 3.5599 (-7)	$k = 4$ 3.573059408099 (-11)	$k = 4$ 6.2803698347350977 (-16)
2.	$k = 1$ 4.14385 (-5)	$k = 2,3$ 4.08539 (-7)	$k = 4$ 2.4465085073184992 (-11)	$k = 4$ 1.9229626863835648 (-16)
5.	$k = 1$ 1.82204 (-5)	$k = 2,3$ 3.02912 (-7)	$k = 4$ 1.2278995694091409 (-11)	$k = 4$ 5.5749338175931272 (-15)
10.	$k = 1$ 7.9958 (-6)	$k = 2,3$ 1.61802 (-7)	$k = 4$ 6.6977253325456931 (-12)	$k = 4$ 9.0213733554231248 (-15)
20.	$k = 1$ 3.168999	$k = 3$ 7.7466 (-8)	$k = 4$ 3.5091008112523597 (-12)	$k = 3$ 8.6490577826669137 (-15)
100.	$k = 1$ 3.13166 (-7)	$k = 3$ 6.765 (-9)	$k = 4$ 7.2371004464164272 (-13)	$k = 3,4$ 6.9735020682421672 (-14)

The results obtained via our proposed schemes (2.2) is very impressive.

Problem 3:

Cash (1981), Ikhile (2001)

$$y_1' = -\alpha y_1 + \beta y_2 + (\alpha + \beta - 1)e^{-t}, \; y_1(0) = 1$$
$$y_2^1 = \beta y_1 + \alpha y_2 + (\alpha - \beta - 1)e^{-t}, \; y_2(0) = 1 \qquad (4.3)$$
$$y_1(t) = y_2(t) = e^{-t}, \; t \in [0, 20], \; \alpha = 1, \; \beta = 15$$

Table 3: Performance of some algorithms on the IVP (4.3)

Best of Cash (1981), $h = 0.1$					**Theoretical solution**
CBD Scheme			**EBD Scheme**		
x	y_1	y_2	y_1	y_2	$[y_1 = y_2]$
5.0	0.64528 (-2)	0.6893465 (-2)	0.673793 (-2)	0.6737977 (-2)	0.673794 (-2)
10.0	0.9031082 (-1)	0.6385428 (-1)	0.4539983 (-4)	0.450011 (-4)	0.4539993 (-4)
20.0	- 21672.865	-1096.9431	0.2061162 (-8)	0.2061154 (-8)	0.2061154 (-8)

Best of Ikhile (2001), $h = 0 \cdot 001$			Best of schemes (2.2), $h = 0 \cdot 001$	
x	y_1	y_2	y_1	y_2
5.0	$k = 1$ 0.6740021 (-2)	$k = 1$ 0.6734833 (-2)	$k = 3$ 0.6737946990853525 (-2)	$k = 4$ 0.67379469990857983 (-2)
10.0	$k = 1,2,3$ 0.4541872 (-4)	$k = 1$ 0.45354207 (-4)	$k = 4$ 0.4539929762494144 (-4)	$k = 4$ 0.45399929762491935 (-4)
20.0	$k = 1$	$k = 1$	$k = 2,3,4$	$k = 2,3,4$

	0. 2061131 (-8)	0.2057011 (-8)	0.20611536224353 (-8)	0.20611536224359 (-8)

	Theoretical solution
x	$[y_1 = y_2]$
5.0	0.67379469990854375 (-2)
10.0	0.45399929762489530 (-4)
20.0	0.20611536224355000 (-8)

The methods (2.2) appear more accurate than existing methods.

5.0 Conclusion

In this presentation, we considered a class of variable order one-step methods for initial value problems in ordinary differential equations. The aim of deriving higher order A or L-stable one-step methods for Equation (1.1) rests on the fact that better accuracy of solutions is desirable. This has been significantly achieved in this paper. The performance of our schemes on the test problems is impressive. Our schemes which are of order $k+2$ perform better than those of Ikhile (2001) which are of order $k+1$..

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