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Effect of Biot number on thermal criticality in a couette flow

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Abstract

This paper studies the effect of Biot number on thermal criticality in a strongly exothermic reaction of a viscous combustible material placed in a channel with lower isothermal fixed wall and upper uniformly moving non-isothermal wall under Arrhenius kinetics, neglecting the consumption of the material. Analytical solutions are constructed for the governing nonlinear boundary-value problem using perturbation technique together with a special type of Hermite-Padé approximants and important properties of the temperature field including bifurcations and thermal criticality are discussed.

Keywords: Couette flows, Arrhenius kinetics, Biot number, Hermite-Padé approximants.

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1.0 Introduction

The problem of hydrodynamic thermal explosions in flows of reacting viscous incompressible fluid is of fundamental importance in many industrial processes in order to ensure safety of life and properties ([2]). In recent time, the theory of the mathematical problem of thermal explosion mainly focus on the determination of critical regimes thought of as regimes separating the regions of explosive and non-explosive ways of chemical reactions ([1], [2], [4]). The classical formulation of this type of problem was first introduced by Frank-Kamenetskii ([4]). Neglecting the reactant consumption, the equation for the heat balance in the original variables together with the boundary conditions can be written as

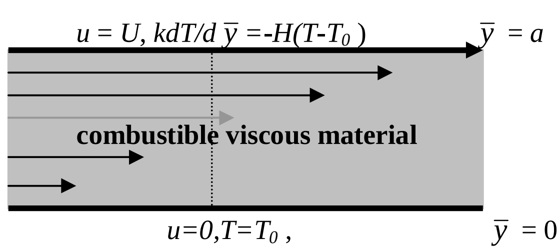

Figure 1: Geometry of the problem

$$\frac{d^2T}{d\bar{y}^2}+\frac{QC_0A}{k}e^{-\frac{E}{RT}}+\frac{\mu}{k}\left(\frac{du}{d\bar{y}}\right)^2=0,\quad \frac{d^2u}{d\bar{y}^2}=0, \tag{1.1a}$$

$$u = U, \quad k\frac{dT}{d\bar{y}} + H(T - T_0) = 0, \text{ on } \bar{y} = a, \tag{1.1b}$$

$$u=0, T = T_0 \text{ on } \bar{y} = 0, \tag{1.1c}$$

where T is the absolute temperature, T_0 the geometry wall temperature, k the thermal conductivity of the material, Q the heat of reaction, A the rate constant, E the activation energy, R the universal gas constant, C_0 the initial concentration of the reactant species, a the channel width, $\bar{y}$ the distance measured in the normal direction, H the heat transfer coefficient and μ is the combustible material dynamic viscosity coefficient, ([1], [2], [4], [5], [6]). Following [4], we introduce the following dimensionless variables in equation (1):

$$\theta = \frac{E(T - T_0)}{RT_0^2}, \quad \varepsilon = \frac{RT_0}{E}, \quad y = \frac{\bar{y}}{a}, \quad \lambda = \frac{QEAa^2C_0 e^{-\frac{E}{RT_0}}}{T_0^2 Rk},$$
$$W = \frac{u}{U}, \beta = \frac{\mu U^2 e^{\frac{E}{RT_0}}}{QAa^2C_0}, Bi = \frac{Ha}{k}, \tag{1.2}$$

and obtain the dimensionless governing equation together with the corresponding boundary conditions as

$$\frac{d^2\theta}{dy^2} + \lambda(e^{\left(\frac{\theta}{1+\varepsilon\theta}\right)} + \beta) = 0, \; \theta(0) = 0, \tag{1.3a}$$

$$\frac{d\theta}{dy} + Bi\,\theta = 0, \text{ on } y = 1, \tag{1.3b}$$

where the fluid velocity profile is given as $W(y) = y$ and $\lambda, \varepsilon, \beta, Bi$ represent the Frank-Kamenetskii parameter, activation energy parameter, the viscous heating parameter and the Biot number respectively. In the following sections, equation (1.3) is solved using both perturbation and multivariate series summation techniques ([5], [6], [7], [8], [9]).

2.0 Perturbation Method

To solve equation (1.3), it is convenient to take a power series expansion in the Frank-Kamenetskii parameter λ, i.e., $\theta = \sum_{i=0}^{\infty} \theta_i \lambda^i$. Substitute the solution series into equation (1.3) and collecting the coefficients of like powers of λ, we obtained and solved the equations governing the coefficients of solution series. The solution for the temperature field is given as

$$\theta(y) = -\frac{\lambda y(\beta+1)}{2(1+Bi)}(y + yBi - Bi - 2) + \frac{\lambda^2 y(\beta+1)}{24(1+Bi)^2}(y^3 + 2y^3Bi + y^3Bi^2$$
$$-4y^2 - 6y^2Bi - 2y^2Bi^2 + Bi^2 + 5Bi + 8) + O(\lambda^3) \tag{2.1}$$

$$Nu = \frac{d\theta}{dy}(y = 0; \lambda, \varepsilon, \beta, Bi)$$

Using computer symbolic algebra package (MAPLE), we obtained the first 30 terms of the above solution series (4) as well as the series for the lower wall heat transfer rate.

3.0 Bifurcation study

The main tool of this paper is a simple technique of series summation based on a special type of Hermite-Padé approximation technique and may be described as follows. Let us suppose that the partial sum

$$U_{N-1}(\lambda)=\sum_{i=0}^{N-1}a_i\lambda^i=U(\lambda)+O(\lambda^N) \quad \text{as } \lambda\to 0, \tag{3.1}$$

is given, we construct a multivariate series expression of the form

$$F_d(\lambda,U_{N-1})=A_{0N}(\lambda)+A_{1N}^d(\lambda)U^{(1)}+A_{2N}^d(\lambda)U^{(2)}+A_{3N}^d(\lambda)U^{(3)}, \tag{3.2}$$

whereby, we substitute in equation (3.2) $U^{(1)}=U$, $U^{(2)}=U^2$, $U^{(3)}=U^3$ for cubic algebraic approximant and $U^{(1)}=U$, $U^{(2)}=DU$, $U^{(3)}=D^2U$, $D=d/d\lambda$ for second order differential approximant, such that

$$A_{0N}(\lambda)=1,\ A_{iN}(\lambda)=\sum_{j=1}^{d+i}b_{ij}\lambda^{j-1}, \text{ and } F_d(\lambda,U)=O(\lambda^{N+1}) \text{ as } \lambda\to 0, \tag{3.3}$$

and $d\geq 1$, i =1, 2, 3. The condition (3.3) normalizes the F_d , reduces the problem to a system of N linear equations for the unknown coefficients of F_d and ensures that the order of series A_{iN} increases as i and d increase in value. We shall take N = 3(2 + d), so that the number of equations equals the number of unknowns. The algebraic approximant enables us to obtain the solution branches while the dominant singularity or criticality in the problem is obtained easily using the differential approximant.

The critical exponent α_N can easily be found by using Newton's polygon algorithm. However, it is well known that, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtains α_N =1/2. If we assume a singularity of algebraic type as in equation (3.2), then the exponent may be approximated by

$$\alpha_N=1-\frac{A_{2N}(\lambda_{CN})}{DA_{3N}(\lambda_{CN})}. \tag{3.4}$$

For details on the above procedure, interested readers can see ([10], [11], [12]).

4.0 Results and discussion

The bifurcation procedure above is applied on the first 30 terms of the solution series and we obtained the results as shown in Tables 1 and 2 below:

Table 1: Computations Showing the Procedure Rapid Convergence for ε = 0.0, β=1.0, $Bi\to\infty$.

d	**N**	Nu	λ_c	α_{cN}
4	18	5.149227914	2.64069774873701	0.4999999
6	24	5.149227922	2.64069775538753	0.5000000
8	30	5.149227922	2.64069775538753	0.5000000

Table 2: Computations Showing Criticality and Wall Heat Transfer for Various values of Parameters (Bi, β, ε)

Bi	β	ε	Nu	λ_c	α_{cN}
∞	1.0	0.1	6.807972736	3.0788455621	0.5000000
∞	1.0	0.0	5.149227922	2.6406977553	0.5000000
1.0	1.0	0.0	3.212645809	1.1138773831	0.5000000
1.0	2.0	0.0	3.699751237	0.9233011827	0.5000000
0.0	1.0	0.0	2.574613961	0.6601744388	0.5000000

Table 1 shows the rapid convergence of the dominant singularity λ_c i.e. the thermal criticality in the flow field together with its corresponding critical exponent α_c and lower wall heat transfer rate *Nu* with gradual increase in the number of series coefficients utilized in the approximants. In Table 2, we observed that the magnitude of thermal explosion criticality at very large activation energy ($\varepsilon = 0$) is lower than that of moderate value of activation energy ($\varepsilon = 0.1$), hence thermal explosion will occur faster in the former than latter. It is interesting to note also in Table 2 that a decrease in the magnitude of Biot number ($Bi \to 0$) as well as an increase in the magnitude of viscous heating parameter β will lower the magnitude of thermal criticality and hence enhance the early occurrence of thermal explosion. Figure 2 shows the fluid temperature profile. A transverse increase in the fluid temperature is observed, however, a further increase in the fluid temperature is noticed with a decrease in Biot number. In Figure 3, a sketch of the bifurcation diagram is presented. Two solution branches (Type I and II) are identified with a bifurcation point at λ_c (i.e. turning point).

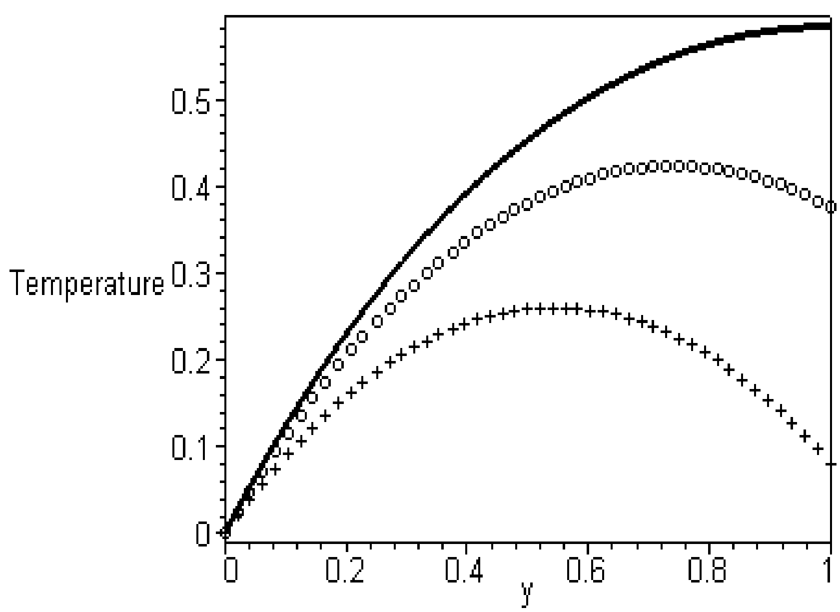

Figure.2. Temperature profile, β=1,λ=1, ε =0,_______ *Bi*=0, *oooooooBi*=1, ++++++*Bi*=10

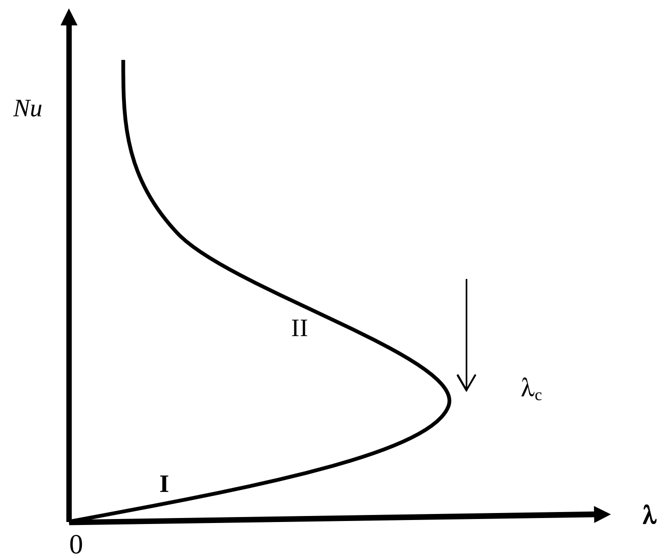

Figure.3. A sketch of bifurcation diagram

5.0 Conclusions

The hydrodynamic thermal explosion in Couette flows is investigated using perturbation series summation and improvement technique. A bifurcation study by analytic continuation of a power series in the bifurcation parameter for a particular solution branch is performed. The procedure reveals accurately the steady state thermal criticality conditions as well as the solution branches. Finally, the above series summation procedure can be used as an effective tool to investigate several other parameter dependent nonlinear boundary-value problems.

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