Journal of the Nigerian Association of Mathematical Physics, Volume 9, (November 2005)

Thermal ignition in a reactive variable viscosity Poiseuille flow

W. Gbolagade
Department of Mathematical Sciences,
Olabisi Onabanjo University, Ago-Iwoye, Ogun State Nigeria
and
O. D. Makinde
Applied Mathematics Department,
University of Limpopo, Private Bag X1106, Sovenga 0727, South Africa

Abstract

In this paper, we investigate the thermal ignition in a strongly exothermic reaction of a variable viscosity combustible material flowing through a channel with isothermal walls under Arrhenius kinetics, neglecting the consumption of the material. Analytical solutions are constructed for the governing nonlinear boundary-value problem using perturbation technique together with a special type of Hermite-Padé approximants and important properties of the temperature field including bifurcations and thermal criticality are discussed

Keywords*:* Poiseuille flow, Thermal ignition, Arrhenius kinetics, Hermite-Padé approximants.

pp 573 - 578

1.0 Introduction

In petrochemical industries as well as petroleum refineries, the study of thermal ignition in a combustible reacting variable viscosity fluid is of great importance in order to ensure safety of life and properties ([3], [9]). Thermal ignitions occur when the reactions produce heat too rapidly for a stable balance between heat production and heat loss to be preserved. Hence, it is important to know the critical values of the basic physical quantities, such as the ambient temperature, surface characteristics, the chemistry of the reacting combustible material and the physical storage geometry at which ignition occur, ([1], [2], [4], [7], [8]). The classical formulation of this type of problem was first introduced by Frank-Kamenetskii ([4]). Neglecting the reactant consumption, the equation for the heat balance in the original variables together with the boundary conditions can be written as

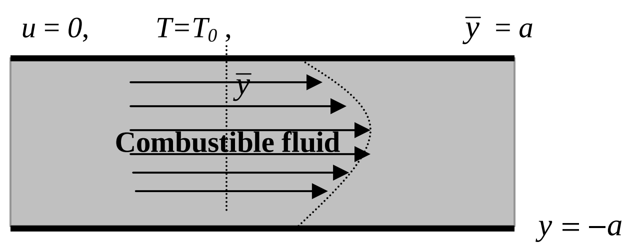

Figure 1: Geometry of the problem

$$\frac{d^2T}{d\bar{y}^2} + \frac{QC_0A}{k}e^{-\frac{E}{RT}} + \frac{\mu}{k}\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\bar{y}}\left(\mu\frac{d\bar{u}}{d\bar{y}}\right) = -G, \tag{1.1a}$$

$$u=0,\ T = T_0,\ \text{on}\ \bar{y} = a \tag{1.1b}$$

$$\frac{dT}{d\bar{y}} = \frac{d\bar{u}}{d\bar{y}} = 0 \ \text{ on } \bar{y} = 0, \tag{1.1c}$$

where T is the absolute temperature, G the constant axial pressure gradient, T_0 the wall reference temperature, k the thermal conductivity of the material, Q the heat of reaction, A the rate constant, E the activation energy, R the universal gas constant, C_0 the initial concentration of the reactant species, a the channel characteristic half width, $(\bar{x}, \bar{y})$ the distance measured in the axial and normal directions respectively. Following Makinde [5], we define the dynamic viscosity of the combustible material as

$$\mu = \mu_0 e^{\frac{E}{RT}}, \tag{1.2}$$

where μ_0 is the combustible material reference viscosity. The following dimensionless variables are introduced into equation (1.1):

$$\theta = \frac{E(T - T_0)}{RT_0^2},\ \ \varepsilon = \frac{RT_0}{E},\ \ y = \frac{\bar{y}}{a},\ \ \lambda = \frac{QEAa^2C_0 e^{-\frac{E}{RT_0}}}{T_0^2 Rk},$$
$$W = \frac{\mu_0 \bar{u} e^{\frac{E}{RT_0}}}{Ga^2}, \beta = \frac{G^2a^2}{QC_0A\mu_0}, \tag{1.3}$$

and obtain the dimensionless governing equation together with the corresponding boundary conditions as

$$\frac{dW}{dy} = -ye^{\left(\frac{\theta}{1+\varepsilon\theta}\right)},\ \ \frac{d^2\theta}{dy^2} + \lambda\left(1+\beta y^2\right)e^{\left(\frac{\theta}{1+\varepsilon\theta}\right)} = 0, \tag{1.4a}$$

$$\frac{d\theta}{dy}(0) = 0, \theta(1) = W(1) = 0, \tag{1.4b}$$

where λ ,ε,β, represent the Frank-Kamenetskii parameter, activation energy parameter and the viscous heating parameter respectively. In the following sections, equation (1.4) is solved using both perturbation and multivariate series summation techniques ([7], [8], [10], [12]).

2.0 Perturbation method

To solve equation (1.4), it is convenient to take a power series expansion in the Frank-Kamenetskii parameter λ, i.e. $\theta = \sum_{i=0}^{\infty} \theta_i \lambda^i$. Substituting the solution series into equation (1.4) and collecting the coefficients of like powers of λ, we obtained and solved the equations governing the coefficients of solution series. The solution for the temperature and velocity fields are given as

$$\theta(y) = -\frac{\lambda(y^2-1)}{12}\left(\beta y^2 + 6 + \beta\right) + \frac{\lambda^2(y^2-1)}{10080}(15y^6\beta^2 + 196\beta y^4 + 15\beta^2 y^4$$
$$-55\beta^2y^2 + 420y^2 - 224\beta y^2 - 644\beta - 2100 - 55\beta^2) + O(\lambda^3) \tag{2.1a}$$
$$W(y) = \frac{1}{2}(y^2-1) - \frac{\lambda}{72}(y^2-1)^2(\beta y^2 + 2\beta + 9)$$

$$+\frac{\lambda^2}{30240}(y^2-1)^2(15y^6\beta^2+231\beta y^4+30\beta^2y^4+42\beta y^2-25\beta^2y^2+840y^2-80\beta^2-777\beta-2100)+O(\lambda^3) \quad (2.1b)$$

Using computer symbolic algebra package (MAPLE), we obtained the first 30 terms of the above solution series (2.1) as well as the series for the lower wall heat transfer rate

$$Nu=\frac{d\theta}{dy}(y=1;\lambda,\varepsilon,\beta).$$

3.0 Bifurcation study

The main tool of this paper is a simple technique of series summation based on a special type of Hermite-Padé approximation technique and may be described as follows. Let us suppose that the partial sum

$$U_{N-1}(\lambda)=\sum_{i=0}^{N-1}a_i\lambda^i=U(\lambda)+O(\lambda^N) \text{ as } \lambda\to 0, \quad (3.1)$$

is given, we construct a multivariate series expression of the form

$$F_d(\lambda,U_{N-1})=A_{0N}(\lambda)+A_{1N}^d(\lambda)U^{(1)}+A_{2N}^d(\lambda)U^{(2)}+A_{3N}^d(\lambda)U^{(3)}, \quad (3.2)$$

whereby, we substitute in equation (7) $U^{(1)}=U$, $U^{(2)}=U^2$, $U^{(3)}=U^3$ for cubic algebraic approximant and $U^{(1)}=U$, $U^{(2)}=DU$, $U^{(3)}=D^2U$, $D=d/d\lambda$ for second order differential approximant, such that

$$A_{0N}(\lambda)=1,\ A_{iN}(\lambda)=\sum_{j=1}^{d+i}b_{ij}\lambda^{j-1}, \text{ and } F_d(\lambda,U)=O(\lambda^{N+1}) \text{ as } \lambda\to 0, \quad (3.3)$$

and $d\geq 1$, i =1, 2, 3. The condition (3.3) normalizes the F_d, reduces the problem to a system of N linear equations for the unknown coefficients of F_d and ensures that the order of series A_{iN} increases as i and d increase in value. We shall take N = 3(2 + d), so that the number of equations equals the number of unknowns. The algebraic approximant enables us to obtain the solution branches while the dominant singularity or criticality in the problem is obtained easily using the differential approximant.

The critical exponent α_N can easily be found by using Newton's polygon algorithm. However, it is well known that, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtain α_N =1/2. If we assume a singularity of algebraic type as in equation (7), then the exponent may be approximated by

$$\alpha_N=1-\frac{A_{2N}(\lambda_{CN})}{DA_{3N}(\lambda_{CN})}. \quad (3.4)$$

For details on the above procedure, interested readers can see ([5], [6], [10], [11], [12]).

4.0 Results and Discussion

The bifurcation procedure above is applied on the first 19 terms of the solution series and we obtained the results as shown in Tables 1 and 2.

Table 1: Computations Showing the Procedure Rapid Convergence for $\varepsilon = 0.0$, $\beta=0.0$

d	**N**	*Nu*	λ_c	α_{cN}
1	9	2.00011662	0.878465558873	0.4999999
2	12	2.00000000	0.878457679761	0.5000000
3	15	2.00000000	0.878457679781	0.5000000
4	18	2.00000000	0.878457679781	0.5000000

Table 2: Computations Showing Thermal Ignition Criticality and Wall Heat Transfer for Various values of Parameter (β), $\varepsilon = 0$.

β	**Nu**	λ_c	α_{cN}
0	2.000000000	0.878457679	0.5000000
1	2.250005680	0.769721022	0.5000000
2	2.451073733	0.683568608	0.5000000
3	2.615047090	0.613901674	0.5000000

Table 1 shows the rapid convergence of the dominant singularity λ_c i.e. the thermal ignition criticality together with its corresponding critical exponent α_c and wall heat transfer rate *Nu* with gradual increase in the number of series coefficients utilized in the approximants. Table 2 shows the magnitude of thermal ignition criticality for combustible material at very large activation energy ($\varepsilon = 0$). It is interesting to note that a decrease in the thermal ignition criticality occurs due to addition of viscous heating, hence, thermal ignition will occur faster in the present of viscous heating. Figures 2 to 3 show both the temperature and the velocity profiles. The fluid temperature increases with an increase in the viscous heating parameter, the same is observed with fluid velocity profile. Two solution branches (type I and II) are identified with a bifurcation point at λ_c (i.e. turning point) as shown in a sketch of bifurcation diagram in Figure 3 below.

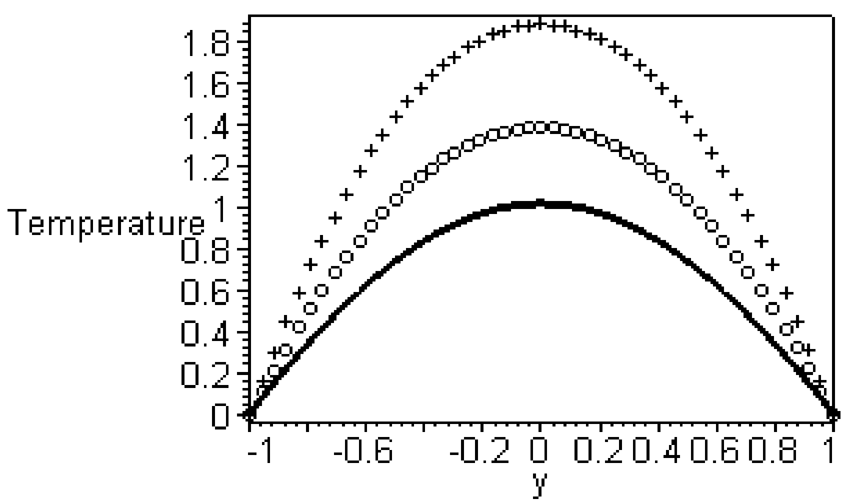

Figure 2. Temperature profile for λ =1; ε=0; β = 0; ooooooooβ = 1; ++++++++β=2.

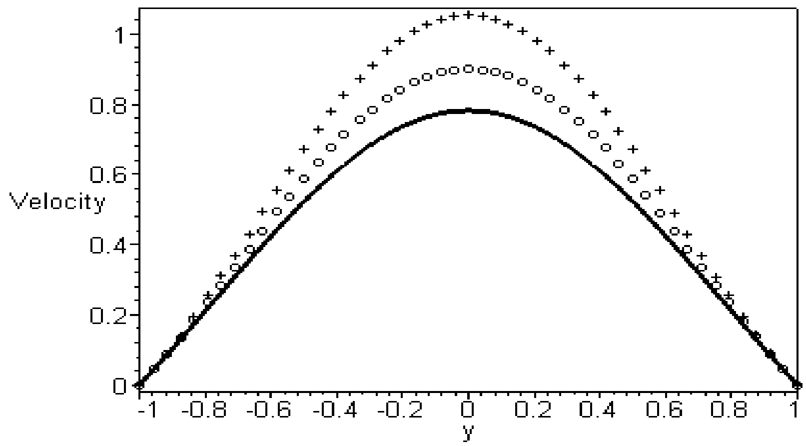

Figure 3. Velocity profile for λ =1; ε=0;________ β=0; oooooooo β=1; ++++++++ β=2.

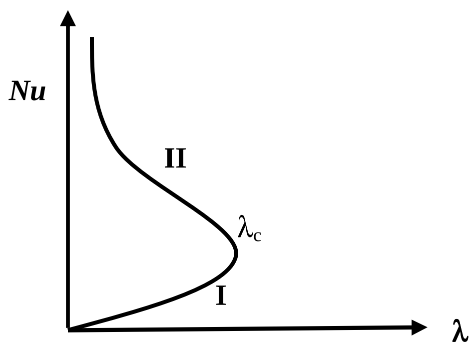

Figure 4. A sketch of bifurcation diagram

5.0 Conclusions

The steady flow of reactive variable viscosity fluid in a channel with isothermal walls is investigated using perturbation series summation and improvement technique. A bifurcation study by analytic continuation of a power series in the bifurcation parameter for a particular solution branch is performed. The procedure reveals accurately the steady state thermal ignition criticality conditions for as well as their dependent on viscous heating parameter. Finally, the above series summation procedure can be used as an effective tool to investigate several other parameter dependent nonlinear boundary-value problems.

References

[1] Balakrishnan, E., Swift, A., Wake, G. C., 1996. Critical values for some non-class A geometries in thermal ignition theory. Mathl. Comput. Modelling 24 (8), 1-10.

[2] Bebernes, J., Eberly, D., 1989. Mathematical problems from combustion theory. Springer-Verlag, New York.

[3] Bowes, P. C., 1984. Self-heating: Evaluating and Controlling the Hazard. Amsterdam: Elsevier.

[4] Frank Kamenetskii, D. A., 1969. Diffusion and Heat Transfer in chemical kinetics. Plenum Press, New York.

[5] Guttamann, A. J., 1989. Asymptotic analysis of power –series expansions, Phase Transitions and Critical Phenomena, C. Domb and J. K. Lebowitz, eds. Academic Press, New York, 1-234.

[6] Hunter, D. L., Baker, G. A., 1979. Methods of series analysis III: Integral approximant methods, Phys. Rev. B 19, 3808-3821.

[7] Makinde, O. D., 2004. Exothermic explosions in a Slab: A case study of series summation technique. International Communications in Heat and Mass Transfer, Vol. 31, No.8, 1227-1231.

[8] Makinde, O. D., 2005. Strong Exothermic explosions in a cylindrical pipe: A case study of series summation technique. Mechanics Research Communications Vol. 32, 191-195.

[9] Makinde, O. D., Osalusi, E., 2005. Second law analysis of Laminar flow in a channel filled with saturated porous media. Entropy, Vol. 7, No. 2, 148-160.

[10] Makinde O. D., Osalusi, E., 2005. Exothermic explosions in symmetric geometries-An exploitation of perturbation technique, Romanian Journal of Physics (formerly Revue Roumaine de Physique) Vol. 50, No. (5-6), 581-584.

[11] Sergeyev, A. V., Goodson, D. Z., 1998. Summation of asymptotic expansions of multiple valued functions using algebraic approximations-application to anharmonic oscillators. J. Phys. A31, 4301-4317.

[12] Tourigny, Y., Drazin, P. G., 2000. The asymptotic behaviour of algebraic approximants. Proc. Roy. Soc. London A456, 1117-1137.