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On actions of algebraic groups

Henry Osaretin Omokaro
Department of Mathematics, University of Benin,Benin City.

Abstract

In [6] ***we discussed the Lie Algebra associated with an algebraic group G. In this work, we employ morphical action of G to obtain a necessary and sufficient condition for a finite dimensional subspace F of K[X] to be stable under all translations where K[X] denotes the set of polynomials in the variables $x_1, x_2, \ldots, x_n$. Group action is discussed briefly as a build up to morphical action of algebraic group.***

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1.0 Introduction

We start this work by looking at group action.

Definition

Let G be a group and *X* a set. Consider a map $G \times X \to X$: $(x,y) \to x.y$ which satisfies the following conditions:

(1) $(x_1x_2)y = x_1(x_2y)$

and

(2) $ey = y$

for all $x_1, x_2 \in G$, $y \in X$, where *e* is the identity element of *G*. We say that *G* acts on *X* on the left. Similarly, we define right group action. In this paper we shall simply call it a group action on *X*. In fact a set, which a group acts on, is called a G-space [3].

Definition

Let a group *G* act on a set X and let $y \in G$. The set $Gy = \{x \in X : xy = y\}$ is called the *isotropy group* of *y*.

The set $G.y = \{xy : x \in G\}$ is called the *orbit* of y. An element *y* of X is called a *fixed point* of the group action if $G.y = y$ [5].

There are numerous examples of group actions in [5]. Group action has been extended to the case of an algebraic group acting on a variety [1].

2.0 Morphical action of algebraic group

Definition

Let $A^n(K)$ denote the affine *n*-space and *K* a field equipped with the Zariski topology. A subset of $A^n(K)$ is called an *affine algebraic set*.

An affine algebraic set is called *irreducible* if whenever V_1 and V_2 are closed subsets of $A^n(k)$ and $X = V_1 \cup V_2$ then either $X = V_1$ or $X = V_2$.

An affine algebraic set X is called an *affine variety* if it is irreducible.

Let *X* and *Y* be varieties. *A* mapping $\varphi: X \to Y$ is called a *morphism of varieties* if it satisfies the following conditions:

1. φ is continuous
2. For every regular mapping, $f: Y \to K$ and every open subset *V* of *Y*, the mapping: $f \circ \varphi: \varphi^{-1}(V_1) \to K$ is a *regular mapping*.

Definition

Let G be a set, which satisfies the following conditions:

1. G is an abstract group
2. G is a topological space with respect to the Zariski topology
3. The group operations G x G $\rightarrow$ G: $(x,y) \rightarrow xy$ and G $\rightarrow$ G: $x \rightarrow x^{-1}$ are morphism where x^{-1} denotes the inverse of x in G. G is called an ***algebraic group*** [6].

One would like to mention here that an algebraic group is not necessarily a topological group except in dimension *O* because in our definition above, *G* x *G* has the Zariski topology whereas in a topological group *G* x *G* has the product topology [5].

Definition

Let G be an algebraic group and X an algebraic variety. If there exists a morphism G x X $\rightarrow$ X such that the following conditions are satisfied:

1. $(x_1x_2)y = x_1(x_2y)$
2. $ey = y \ \forall \ x_1, x_2 \in G$, $y \in X$, where *e* is the identity element of G, we say that *G acts morphically on the variety X.*

Let us now consider some examples of morphical group action. In the following examples, G is an algebraic group.

1. The mapping ϕ:ln(G) x G $\rightarrow$ G such that $\phi(f_x, g) = f_x g \ \forall \ f_x \in \ln(G)$, $g \in G$ where ln(*G*) denotes the set of all inner authomorphisms of G [5] is a morphical action of G on itself.
2. The mapping $\phi: G_L \times G \rightarrow G$ defined by $\phi(L_x, g) = L_x g \ \forall \ L_x \in G_L$, $g \in G$ is a morphical action of G on itself as left translation. We know that ϕ is a group action [5]. In fact it is a morphical action of the algebraic group *G* on itself.
3. Let $\phi: G \rightarrow GL(V)$ be a rational representation of *G*. The operation $G \ x \ V \rightarrow V$ such that $x.v = \phi(x)v$ is a morphical group action of *G* on *V*, where *V* is a vector space over the field associated with the algebraic group *G*. Clearly, $x_1x_2(V) = \phi\ (x_1x_2)\ V = \phi(x_1)\ \phi(x_2)V = \phi(x_1)\ (\phi(x_2)V) = x_1(x_2V) \ \forall \ x_1, x_2 \in G$, $v \in V$.

Also $ev = \phi(e)v = v$, since $\phi(e)$ is the identity linear transformation, ϕ being a homomorphism. Therefore *G* acts on *V*. In fact it is a morphical action of *G*. It is easy to see that *V* forms a G-module over G with respect to the operation $x.v = \phi(x)v \ \forall x \in G$, $v \in V$. The G-module *V* is called a *rational G-module.*

Now let ϕ: $G \rightarrow GL(V)$ be a rational representation of the algebraic group *G*. If we identify G with the affine n-space (*n*-dim(*V*)) it is clear that the operation $xv = \varphi(x)\ (v)$, $(x \in G, v \in V)$ define an action of *G* on *V*. The map ϕ: $G \rightarrow GL(V^*)$ such that $\phi(x)f = x.f$, where V^* is the dual vector space of *V*, is called the *dual or contragradient representation* [5].

Note that $(x.f)\ (v) = f(x^{-1}v)$ and x^{-1} is written to ensure that $y(x.f) = (yx)f \ \forall \ f \in V^*$, $v \in V$, $x \in G$.

Now let $v_1, v_2 \in V$. Then

$$\begin{aligned}(x.f)\ (v_1+v_2) &= f(x^{-1}(v_1+v_2)), \text{ by definition} \\ &= f(x^{-1}v_1 + x^{-1}v_2), \text{ V is a G-module} \\ &= f(x^{-1}v_1) + f(x^{-1}v_2), f \text{ being linear} \\ &= (xf)(v_1) + (xf)(v_2), \text{ by definition.}\end{aligned}$$

3.0 Morphisms obtained by translations

In this section, we discuss linear actions of an algebraic group on an affine algebra *K*[*X*] and some of its finite dimensional spaces when G acts on an affine variety *X* where *K*[*X*] denotes the set of polynomials in the variables $x_1, x_2, \ldots, x_n$.

If an algebraic group G acts morphically on an affine variety *X* we know that $G \times G \rightarrow G$:

$(x,y) \to xy$ is a morphism of varieties. It follows that the map $f: X \to X$ such that $f(y) = \phi(x^{-1}, y)(x \in G, y \in X)$ is a morphism of varieties, where ϕ is the group action of the algebraic group G on the variety X. Let us denote by τ_x the comorphism associated with the morphism f for each fixed $x \in$ G and for each $f \in K[X]$ and $y \in X$, consider the operation $(\tau_x f)(y) = f(x^{-1}y)$.

Consider the mapping τ: G $\to$ GL (K(X)) such that $\tau(x) = \tau_x$. Let $g_1, g_2 \in$ G. Then $\tau(g_1 g_2) = \tau_{g_1 g_2}$ by definition. If $f \in K[X]$, consider

$$(\tau(g_1g_2)f)(y) = f(\tau(g_1g_2)(y)) = f((g_1g_2)^{-1}y)$$
$$= f(g^{-1}{}_2 g_1^{-1} y) = f(g_2^{-1}(g_1^{-1}y)) = (\tau(g_2)\,\tau(g_1)f)(y).$$

$\therefore$ τ is a group homomorphism. So τ is a representation of the algebraic group G.

Definition

$\tau(x) = \tau_x$ is called a *Translation of Functions by* ***x***. In fact it is a *K*-algebra authomorphism of K[x].

Now we have considered the action of G on a variety *X*. Suppose, in particular, the variety X = G, so that G acts on itself by left translations $y \to$ xy and by right translations $y \to yx^{-1}$.

For the left translations, the morphism introduced earlier becomes $y \to x^{-1}y$ while for right translations, the mapping becomes $y \to yx$. The comorphism associated with the first one is λ_x while the one associated with the second one becomes ρ_x.λ_x is called *Left Translation of Functions by x* while ρ_x is called *Right Translation of Functions by x***,** where $(\lambda_x f)(y) = f(x^{-1}y)$ and $(\rho_x f)(y) = f(yx)$. So we see that the operations λ: G $y \to GL(K[G])$ and ρ: G $\to$ GL(K[G]), where $\lambda(x) = \lambda_x$ and $\rho(x) = \rho_x$ are both group morphisms.

Proposition

λ_x and ρ_y commutes for each pair of elements $x, y \in$ G.

Proof

We show that $\lambda_x\rho_y = \rho_y\lambda_x$ $\forall x, y \in$ G.

Let $f \in K[G]$: Since $X = G$, $z \in G$ gives

$$[(\lambda_x\rho_y)f](z) = (\lambda_x(\rho_y f))z) = (\lambda_x f)(zy) = f(x^{-1}zy).$$

Similarly,

$$(\rho_y\lambda_x)f)(z) = (\rho_y(\lambda_x f))(z) = (\rho_y f)(x^{-1}z) = f(x^{-1}zy).$$

Therefore $((\lambda_x\rho_y)f)(z) = ((\rho_y\lambda_x)f)(z)$ $\forall z \in G$.

Hence, we conclude that $\lambda_x\rho_y = \rho_y\lambda_x$ for each pair of elements $x, y \in G$. That is λ_x and ρ_y commutes.

We now employ this result, among other things, to obtain a characterization of memberships of closed subgroups in an algebraic group.

Theorem

Let H be a closed subgroup of an algebraic group G, I the ideal of K[G] vanishing on H. Then $H = \{x \in G: \rho_x(I) \subset I\}$.

Proof

First we show that $H \subset \{x \in G: \rho_x(I) \subset I\}$. So let $x \in H$. Our task is to show that $\rho_x(I) \subset I$. So we choose $f \in I$.

Now $(\rho_x f)(y) = f(\rho_x(y)) = f(yx)$ $\forall y \in$ H. Now $x, y \in H \Rightarrow xy \in H$, H being a subgroup. Since $f \in I$, it follows that $f(xy) = 0$. That is $\rho_x f(y) = 0$ $\forall y \in H$, hence $\rho_x f \in$ I.

Conversely, we show that if $x \in$ G such that $\rho_x(I) \subset I$, then $x \in$ H. Let $f \in$ I then $\rho_x(f) \in I$ by assumption and so $(\rho_x f)(z) = 0$ $\forall z \in H$. In particular, $e \in$ H (where e is the identity element of G). Therefore $(\rho_x f)(e) = 0$. But $(\rho_x f)(e) = f(ex) = f(x)$.

Therefore $f(x) = 0$ $\forall f \in$ I and so $x \in$ H. Thus $\{x \in G: \rho_x I \subset I\} \subset H$.

We therefore conclude that $H = \{x \in G: \rho_x I \subset I\}$ and the proof is complete.

Preposition

(a) Let an algebraic group G act morphically on an affine variety X and let F be a finite dimensional subspace of K[X]. There exists a finite dimensional subspace of K[X] including F which is stable under all translations $\tau_x (x \in G)$.

(b) F is itself stable under all translations $\tau_x (x \in G)$ if and only if $\phi^*(F) \subset K[G] \underset{K}{\otimes} F$ where ϕ: G x $X \rightarrow X$ gives the action n of G on X.

Proof

(a) We assume without loss of generality that F is the span of a single $f \in K[X]$ and "add up" the resulting spaces E afterwards. Let us write

$\phi^* f = \sum f_i \otimes g_i \in K[G] \otimes K[X]$., For each $x \in G$, $y \in X$, $(\tau_x f)(y) = f(x^{-1}y) = \sum f_i(x^{-1}) g_i(y)$

where $\tau_x f = \sum f_i(x^{-1}) g_i$. The functions g_i therefore span a finite dimensional subspace of K[X] which contains all translates of f. So the space E spanned by all $\tau_x f$ gives the desired result.

(b) Suppose $\phi^*(f) \subset K[G] \otimes F$. We show that F is stable under all $\tau_x (x \in G)$. From (a) above, since $\phi^*(F) \subset k[G] \underset{K}{\otimes} F$ we can take the functions g_i to be in F, that is, F is stable under all $\tau_x (x \in G)$ and that proves the first part. Conversely, suppose F is stable under all τ_x we show that $\phi^*(F) \subset k[G] \otimes F$.

We can extend a vector space basis f_i of F to a basis $f_i \cup g_i$ of $K[X]$. If $\phi^*(F) = \sum r_i \otimes f_i + \sum s_i \otimes g_i$ we have $\tau_x f = \sum r_i \otimes f_i + \sum s_i(x^{-1}) g_i$

Clearly, the functions s_i must vanish identically on G. That is $\phi^*(F) \subset k[G] \otimes F$.

References

1. Armand Borel Linear Algebraic Groups, Springer-Verlag, New York, 1985.
2. Jeane Pierre Serre Algebraic Groups and Class Fields, Springer-Verlag, New York, 1995.
3. Thomas Banachoff, John Werner Linear Algebra Through Geometry, Springer-Verlag, New York, 1980.
4. Williams C. Waterhouse Introduction to Affine Group Schemes, Springer-Verlag, New York, 1979.
5. Omokaro, H. O:Action of Algebraic Groups, (ICTP), Trieste, Italy, 1996.
6. Omokaro, H. O, On the Lie Algebra of an Algebraic Group, Journal of the Nigerian Association of Mathematical Physics Vol. 7 (2003) pp 23-26.