Journal of the Nigerian Association of Mathematical Physics, Volume 9 (November 2005)

The frequency equation to determine the eigenvalue of a clamped-clamped uniform Timoshenko beam

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Abstract

This paper presents the solution of frequency equation to clamped ends Timoshenko beam. The Eigenvalue were obtained from the asymptotic formulas.

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1.0 Introduction

When a structural beam is acted upon by an applied force, it is always observed that the frequency of this applied force varied as a result of the variation in the response, the maximum frequency obtained in this case is called the natural frequency of the beams. Therefore the study of natural frequencies on the Timoshenko beam has received considerable attention in recent years for its potential application in engineering devices such as bending stiffness, vibration experiments, and model of Aircraft wings.

This problem has attracted several authors [1-10] who have made significant contribution to this study. Jacquot and Gibson [3] considered the effects of discrete masses and elastic supports on the vibrations of a beam. They obtained the natural frequencies by expanding the mode shape in terms of well-known characteristics beam function in a form of Galerkin's method.

For works in natural frequencies of vibration of beams, mention will be made of Laura et al, [4] used Rayleigh-Ritz method to study the vibration of beams carrying several particles and subjected to an axial force, Liu and Yel [10] used Rayleigh-Ritz method to obtain natural frequency of various types of non-uniform beams.

Despite the importance of vibration of beams, very little has been done on the natural frequencies on the Timoshenko beam. Recently Bruce and Joyce [1] studied in detail the Eigenvalue formula for the uniform Timoshenko beam, the free-free problem. Bruce [2] established and presented asymptotic formula for this Eigenvalue of a free-free uniform Timoshenko beam.

In the present paper, the main objective is to make an investigation of the frequency equation for a clamped ends Timoshenko beam and to solve the two coupled differential equations for the deflection of Timoshenko beam subject to the relevant boundary conditions.

2.0 Governing equation

The governing equation describing the transverse vibration of the beams was first derived by Timoshenko in (1920) [12] and he obtained the coupled partial differential equations stated below.

$$EI\varphi_{xx} + KAGV_x - KAG\varphi - \rho I\varphi_{tt} = 0 \qquad (2.1)$$

$$KAGV_{xx} + KAG\varphi_x - \rho AV_{tt} = F(x,t) \qquad (2.2)$$

where E is the young modulus of elasticity in tension and compression, G is the shear modulus, I is the cross-sectional area inertia of the beam, A is the cross-sectional area of the beam. K is the shear coefficient, $\varphi = \varphi(x,t)$ is the slope of the beam due to bending, while $V = V(x, t)$ is the lateral displacement at time t of cross section located x units from one end of the beam, ρ is the mass density of the beam per unit length and F(x, t) is an applied force.

Assume that the beam rest on the clamped-clamped ends then V and φ must satisfy the following boundary conditions.

$$V_x - \varphi\Big|_{x=0} = 0\ , \quad \varphi_x\Big|_{x=0} = 0 \tag{2.3a}$$

$$V_{xxx} - \varphi_{xx}\Big|_{x=L} = 0\ , \quad \varphi_{xxx}\Big|_{x=L} = 0 \tag{2.3b}$$

Separation of variables argument were used and it was discovered that the Timoshenko equation for V and φ leads to a coupled system of second order differential equations for y and ψ

$$EI\psi_{xx} + KAGy_x - KAG\psi - \rho I\psi_{tt} = 0 \tag{2.4}$$

$$KAGy_{xx} + KAG\psi_x - \rho Ay = 0 \tag{2.5}$$

where p^2 is considered as Eigen parameter. This imply that y and ψ satisfy the same boundary conditions for clamped ends stated in equation (2.3) for V and φ

$$y_x - \psi\Big|_{x=0} = 0, \quad \psi_x\Big|_{x=0} = 0 \tag{2.6a}$$

$$y_{xxx} - \psi_{xx}\Big|_{x=L} = 0, \quad \psi_{xxx}\Big|_{x=L} = 0 \qquad (2.6b\}$$

The self-adjoint boundary problem for y and ψ shows the values of p^2 for which nontrivial solution to this problem exist. Thus the eigenvalues are real and it can be shown that the collection of all eigenvalues for this problem forms a discrete, countable and unbounded set of real nonnegative numbers if ω is the natural frequency. For a beam $p^2 = (2\pi\omega)^2$ is one of the beam's eigenvalues; therefore it is possible to determine the Eigenvalue for natural frequency in an experiment.

Assume that a set of natural frequencies for the beam with unknown elastic moduli and mass density have been determined from vibration experiment and the sequence of eigenvalues have been constructed from the data obtained [1], then we need to know the effect of eigenvalues on these unknown material parameter. To answer the question the dependence of eigenvalues on E, I, kG, A and ρ must be determined. This is not easy because the dependence of the eigenvalues on these coefficients is nonlinear hence the derivation of eigenvalues problems for uniform Timoshenko beam must first be for beams with no constant materials and geometric parameter for a beam to be uniform, then E, I, kG, A and ρ are constant. Due to the density and constant materials, it is very difficult to establish the asymptotic formulas for the Eigenvalues for the clamped ends beams. This paper presents an asymptotic formula for Eigenvalue for uniform clamped ends Timoshenko beam.

3.0 Transcendental equation

Following the assumption that E, I, kG, A and ρ are constant and Huang [3] the transcendal equation is derived in this section with the boundary value problem stated in equation (2.2). Equations (2.3), (2.4) and (2.5) may be dimensionalised in a simplified form.

Let $\zeta = \dfrac{x}{L}, \quad q^2 = \dfrac{\rho AL^4 p^2}{EI}, \quad r^2 = \dfrac{L}{AL^2}, \quad s^2 = \dfrac{EI}{KAGL^2}$

Equations (2.2) and (2.3) can be proved equivalent to the differential equations

$$s^2\psi'' - \left(1 - q^2 r^2 s^2\right)\psi' + \frac{y'}{L} = 0 \qquad (3.1)$$

and

$$y'' - q^2 s^2 y' + L\psi = 0 \qquad (3.2)$$

where (′) denote the differentials with respect to ζ

Similarly the boundary conditions given in (2.4) can be written as

$$y' - \psi\Big|_{\zeta=0} = 0, \quad \psi'\Big|_{\zeta=0} = 0 \qquad (3.3a)$$

$$y''' - \psi''\Big|_{\zeta=1} = 0, \quad \psi'''\Big|_{\zeta=1} = 0 \qquad (3.3b)$$

By eliminating y or ψ from (2.5), or (2.6) it was discovered that these two second order equations imply that y or ψ must satisfy two decoupled fourth order ordinary differential equations

$$y^{iv} - q^2\left(r^2 + s^2\right)y'' - q^2\left(1 - q^2 r^2 s^2\right)y = 0 \qquad (3.4)$$

$$\psi^{iv} - q^2\left(r^2 + s^2\right)\psi'' - q^2\left(1 - q^2 r^2 s^2\right)\psi = 0 \qquad (3.5)$$

Coupling between y and ψ still occur through the boundary conditions (3.1). Define α and β as

$$\alpha = \left[\frac{r^2+s^2}{2} - \sqrt{\frac{r^2-s^2}{2} + \frac{1}{q}}\right]^{\frac{1}{2}} \qquad (3.6)$$

$$\beta = \left[\frac{r^2+s^2}{2} + \sqrt{\frac{r^2-s^2}{2} + \frac{1}{q}}\right]^{\frac{1}{2}} \qquad (3.7)$$

In [3], Huang derives general solution to equations (3.4) and (3.5), valid when $q^2 r^2 s^2$ is not 1 or 0. These solutions are

$$y = c_1 \cos q\alpha\zeta + c_2 \sin q\alpha\zeta + c_3 \cos q\beta\zeta + c_4 \sin q\beta\zeta \qquad (3.8)$$

$$\psi = d_1 \sin q\alpha\zeta + d_2 \cos q\alpha\zeta + d_3 \sin q\beta\zeta + d_4 \cos q\beta\zeta \qquad (3.9)$$

By substituting the general solutions for y or ψ into the second order equations (2.5) and (2.6) the constant c_i may be obtained in terms of the d_i. Thus y can written in terms of the d_i provided $q \neq 0 \, or \, \frac{1}{rs}$, it can be shown that solution to boundary value problem in (3.1) and (3.2) exist if and only if $\overline{A}d = 0$, where $d = (d_1, d_2, d_3, d_4)$ and

$$\bar{A} = \begin{vmatrix} q\alpha & 0 & q\beta & 0 \\ 0 & \dfrac{s^2}{\alpha^2 - s^s} & 0 & \dfrac{s^2}{\beta^2 - s^2} \\ -(q\alpha)^3 \cos q\alpha & -(q\alpha)^3 \sin q\alpha & -(q\beta)^3 \cos q\beta & -(q\beta)^3 \cos q\beta \\ \dfrac{(q\alpha s)^2 \sin q\alpha}{\alpha^2 - s^2} & \dfrac{(q\alpha s)^2 \cos q\alpha}{\alpha^2 - s^2} & \dfrac{(q\beta s)^2 \sin q\beta}{\beta^2 - s^2} & \dfrac{(q\beta s)^2 \cos q\beta}{\beta^2 - s^2} \end{vmatrix} \tag{3.10}$$

***Theorem* 3.1**

Let $\eta = \dfrac{E}{kG} = \dfrac{s^2}{r^2}$ *and* $\chi = \dfrac{4\eta}{(1+\eta)^2}$.

Then a given beam will have a nonzero double Eigenvalue if and only if there exist positive integers k_1 *and* k_2 *such that*

$$r^2 = \frac{\dfrac{2}{\left[(1+\eta)\pi^2\right]}}{\left([k_1 + k_2]^2 + k_2^2\right)\left(\chi - \left(\dfrac{\left([k_1 + k_2]^2 - k_2^2\right)^2}{\left([k_1 + k_2]^2 + k_2^2\right)^2}\right)\right)} \tag{3.11}$$

A double Eigenvalue occurs when the scaled Eigenvalue parameter $q^2 = \left(\dfrac{\rho AL^4}{EI}\right)p^2$ *satisfies*

$$q^2 = \left\{\left([k_1 + k_2]^2 + k_2^2\right)^2 - \left([k_1 + k_2]^2 - k_2^2\right)^2\right\} \tag{3.12}$$

Proof

The matrix *A* is row equivalent to

$$\begin{vmatrix} 1 & 0 & \dfrac{1}{\lambda} & 0 \\ 0 & 1 & 0 & \dfrac{1}{\xi} \\ 0 & 0 & (q\alpha)^2\left(\dfrac{1}{\lambda^2}\cos q\beta - \cos q\alpha\right) & (q\beta)^2\left(\sin q\beta - \dfrac{\lambda^2}{\xi}\sin q\alpha\right) \\ 0 & 0 & (q\beta)^2\left(\dfrac{1}{\xi}\sin q\beta - \sin q\alpha\right) & \dfrac{(q\alpha)^2}{\xi}\left(\dfrac{1}{\lambda^2}\cos q\beta - \cos q\alpha\right) \end{vmatrix} \tag{3.13}$$

where $\xi = \dfrac{\beta^2 - s^2}{\alpha^2 - s^s}$ *and* $\lambda = \dfrac{\alpha}{\beta}$

The double Eigenvalue can be obtained if and only if the null space of this matrix is of two- dimension. And this is possible if and only if the entries in the bottom right 2 x 2 submatrix are equal to zero, which can happen if and only if for some positive integers k_1 *and* k_2, $q\alpha = k_1\pi$ and $q\beta = 2k_2\pi$. Equation (3.11) and (3,12) are equivalent to these two equations $q\alpha$ and $q\beta$.

4.0 Numerical examples of beams with double eigenvalue

Consider a beam whose cross-section is of annulus shape with thickness t and the outer radius $\sigma + \frac{t}{2}$ and the poisons ratio υ is taken as $\frac{1}{4}$, the moment of inertia *j* is

$$j = \int_{r=\sigma-t/2}^{r=\sigma+t/2} \int_{\theta=0}^{\theta=2\pi} (y\cos\theta)^2 y\,dy\,d\theta = \pi\left(\sigma^3 t + \frac{\sigma t^3}{4}\right)$$

The cross-sectional area is $A = 2\pi\sigma t$. by applying a formula in [10], the value of *k* can be written in terms of poison's ratio as $k = \frac{2(1+\upsilon)}{4+3\upsilon} = \frac{10}{19}$. $\frac{E}{G}$ Can be expressed in terms of υ from the existing formula

$$\frac{E}{G} = 2(1+\upsilon) = \frac{5}{2} \qquad (4.1)$$

Thus the values of η *and* χ defined in theorem above were calculated to be $\eta = 4.75$ and $\chi = \frac{304}{529}$ and also from (2.6a) the expression for *I* and *A*,, $r^2 = \frac{I}{AL} = \frac{\sigma^2/2 + t^2/8}{L^2}$. Assume that the thickness *t* of the beam is $\frac{\sigma}{60}$ so that r^2 is written in terms of σ and *L*

$$r^2 = \frac{14401\sigma^2}{22800L^2} \qquad (4.2)$$

The value of $\frac{\sigma}{L}$ is chosen so that the positive integers (k_1, k_2) when substituted into the right-hand side of (4.2) is equal to the right hand of equation (3.9). Let $k_1 = 3$ and $k_2 = 5$. Hence the right hand side of equation (3.9) is

$$\frac{\dfrac{2}{\left[(1+\eta)\pi^2\right]}}{\left([k_1+k_2]^2 + k_2^2\right)\left(\chi - \left(\dfrac{\left([k_1+k_2]^2 - k_2^2\right)^2}{\left([k_1+k_2]^2 + k_2^2\right)^2}\right)\right)} = \frac{16376}{1603375\pi^2}$$

Suppose that σ and *L* are chosen so that $r^2 = \frac{14401\sigma^2}{22800L^2} = \frac{16376}{1603375\pi^2}$. Then

$$\frac{\sigma}{L} = \sqrt{\frac{471628800}{14401*1603375\pi^2}} = 0.04549$$

And from the theorem the beam will have a double Eigenvalue when

$$\left(\frac{46.702.104.\sigma^2\rho}{E}\right)p^2 = \left(\frac{\rho AL^4}{EI}\right)p^2 = q^2$$

$$= \pi^4\left\{\left([k_1+k_2]^2+k_2^2\right)^2 - \left([k_1+k_2]^2-k_2^2\right)^2\right\} = \frac{1603375\pi^4}{16376} \quad (4.3)$$

The sigma must be chosen so that equation (4.3) is satisfied. Also the uniform mass density ρ is assumed to be any positive value and modulus f elastic E is chosen to satisfy the equation (4.1). The determinant of equation (3.7) written as

$$\left|\bar{A}\right| = 1 - \cos q\alpha \cos \beta + {}^1\!/_q \frac{\left(q^2r^2s^2-1\right)^{\frac{1}{2}}\left(q^2\left(r^2-s^2\right)+2\right)\sin q\alpha \sin q\beta}{r^2+s^2}$$

is defined as the frequency function for the resonant frequencies of the clamped ends uniform Timoshenko beam.

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