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Mathematical model for computation of gravity adjusted observations.

M. O. Aku
Department of Physics, Bayero University, Kano - Nigeria

Abstract

The discussion focuses on the method of least squares and various tests that can be performed to validate the results of the adjustment of gravity observations. A system of linear equations was considered for the mathematical model to figure out the relation between the observables (observed quantities) and the parameters (wanted parameters). Standard deviations and variances associated with the observations, and the covariances between the observations were used to build up the weight matrix. A test of the model was performed using relative gravity measurements at stations in northwestern Nigeria. The adjusted results in the estimated values of gravity for all stations, together with their accuracy estimates, tally with known absolute gravity values in the area.

Keywords: *Station, observation, parameters, least-squares, adjustment*

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1.0 Introduction

A least-square adjustment of survey observations is an important step in a gravimetric survey. Properly used, it helps isolate blunders in the observations being adjusted and gives the accuracy and reliability of the gravity values being determined. The primary components of a least-square adjustment are the survey observations (in this case gravity differences) and the uncertainties associated. Due to measurement limitations of the surveying instruments and the influence of the operators, these observations include some level of error. These errors cause loops not to close perfectly and result in different computed values for the same station in the network (Pennington, 1965) [4].

The ultimate goal of a least-square adjustment is to produce a set of observations where all loops close perfectly and only one value can be computed for any point in the network. In order to accomplish this task, the observations going into the adjustment must be changed slightly, i.e. adjusted. Of course we do not want the observations to be modified too much, since this is what was physically observed in the field. But the observations do contain some level of error. Any error associated with an observation is predictable because of the measurement accuracy of the instruments used. A successful adjustment is one where observations are changed as little as possible, and the amount of adjustment to any observation is within expected levels.

Unfortunately there are a number of obstacles that can stand in the way of producing a successful adjustment. Primary on this list are blunders, errors in the observation due to equipment malfunction or operator error (incorrectly measured instrument height, insufficient data, wrong station identifier, etc.). Tools exist to assist in overcoming these obstacles, both before and during the adjustment. The analysis tools are mostly statistically based, as a result, it is very important that uncertainties (error estimates) are realistic. At times, these uncertainties may be little optimistic (too small) or pessimistic (too large). Methods exist to help identify when uncertainties are unrealistic and to help rectify this situation. Also, adjustment analysis tools cannot function properly without redundancy in the observations. This must be borne in mind when designing a survey network. A certain percentage of points should receive multiple observations.

2.0 The mathematical model

Consider the system of linear equations f (X, L), or in, matrix form:

$$AX = L \qquad (2.1)$$

where X is the unknown vector, L is the constant vector, and A is the coefficient matrix or design matrix. Let's assume that the elements of L are the results of physical measurements, L is called the observation vector.

In the case where there are no redundant equations (minimum number of measurements), A is square and non-singular, and therefore has an inverse. The unique solution is then given by $X = A^{-1}L$. When there are redundant equations, the system is over determined: A is not square, but A^TA is, (Mikkhail and Ackerman, 2000) [2] and the mathematics tells us that

$$X = (A^TA)^{-1}A^TL$$

This is so if and only if the system is consistent. But if there are redundant measurements, they will be inconsistent because physical measurements are never perfect. We have to assume that there were errors in the determination of the observations. This leads us into the theory of errors and statistics**.** An adjustment becomes necessary when the data available exceed the minimum required for unique determination.

We must first get rid of systematic errors (for instance the lengthening of the gravimeter spring) and of blunders (gross errors). The way for accounting for systematic errors and intercepting blunders are numerous and we are not going to venture into this here. Even after eliminating those errors, we still have an unavoidable spread in the observations, and we say that the observations contain random or accidental errors. To account for this we have to refer to statistical concepts.

No unique solution will exist, and all we are able to do is make a unique estimate of the solution. The most commonly used criterion for the estimate to be unique is the least squares criterion; that the sum of the squares of the inconsistencies be minimum. Using statistics, we are also usually able to establish the degree of reliability of the solution, and thus define the most probable unique solution.

To cancel the inconsistencies, we add a vector to equation (2.1), which becomes:

$$AX - L = V \qquad (2.2)$$

where V is usually called the residual vector (observation errors). The elements of **V** are not known and must be solved for. So we have to allow some of or all the elements of **L** change slightly while solving for X, or regard L as an appropriate value of some other value $\hat{L}$ which yields the unique solution $\hat{X}$. Now the least squares criterion states that the best estimate $\hat{X}$ for X is the estimate, which will minimize the sum of the squares of the residuals (discrepancies between observations and estimated values assigned to each observable), that is $\hat{V}^T\hat{V}$ is minimum.

The estimated $\hat{X}$ so determined is the least squares estimate. The difference between the observed value and any arbitrarily assumed or computed value is called the misclosure, which is different from the residual (uniquely determined by the difference between the observation and the sample mean).

The limitation to this method is that we have to assume that the parameters are mutually independent, and to postulate a normal probability distribution function for the random errors.

Often the physical measurements which make up the elements of L do not all have the same precision (they have been made using different instruments by different people, under different conditions etc.). This fact should be reflected in our least squares estimation process, so we assign to each measurement a known weight and call P the matrix whose elements are these weights, the weight matrix. We modify the criterion, which becomes $\hat{V}^T P\hat{V}$ is minimum. The resulting estimate is called the weighted least squares estimate, and is given by

$$\hat{X} = (A^T PA)^{-1} A^T PL \qquad (2.3)$$

where $A^T PA$ is the normal equation matrix, and must not be singular for the estimator to be unique.

Further complications of the mathematical model arise when the functions involved are non-linear, but this problem will not be discussed here. From a practical point of view, the inversion and multiplication of large matrices require a considerable number of computation steps. The use of fast computers alone allows us to calculate the solutions for large systems of equation where before those machines were widespread in use, the task would have been attempted only when absolutely necessary.

Let's assume there are n observations and u unknown parameters. The least squares estimation process is applied only when there are redundant measurements, that is, $n > u$. The number $(n - u)$ is called the redundancy or number of degrees of freedom**.**

So far, we have not specified how the weight matrix should be chosen. We are going to use the standard deviations and variances associated with the observations, and the covariances between the observations, to build up this matrix. The variance of an observation is larger when it is less accurately determined. In combining observations, more importance should be attached to those having smaller variances. One reasonable choice for the weight matrix is thus:

$$P = \sum\nolimits_L^{=1} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots \\ \sigma_{21} & \sigma_2^2 & \\ \dots & & \sigma_n^2 \end{bmatrix}^{-1}$$

in this case, values must be assigned to the variances and covariances in the matrix before a least squares estimation can be made. The values arise from knowledge of the measuring instruments and procedures being used. The covariance of a pair of observations is a measure of statistical dependence of the two values. In practice, the covariances are often assumed to be null in gravimetry because the observation values are uncorrelated, but for instance in Global Positioning systems networks adjustments, full variance-covariance matrices are often used.

It is often possible only to assign relative values among the variances and covariances, so that we know Σ_L only to within a scale factor, that is if $\Sigma_L = \sigma_0^2 Q$, we know the relative covariance matrix Q, but not the variance factor σ_0^2. However we can show that in equation (2.3) the variance factor drops out and either weights $\sum_L^{-1}$ or Q^{-1} result in the same estimator.

The weight matrix is thus chosen to be proportional to the inverse of the estimated covariance matrix of the observations. If L is postulated to be uncorrelated, it will be a diagonal matrix. As shown in equation (2.3), the factor for computing P will not influence the result $\hat{X}$**.** The weight matrix accounts for the fact that the data may be of varying quality. Otherwise we would replace the weight matrix with the identity matrix. It can be shown that the least squares unbiased estimator $\hat{\sigma}_0^2$ of the variance factor σ_0^2 is $\qquad \hat{\sigma}_0^2 = \dfrac{\hat{V}^T P\hat{V}}{n-u}$

where $P = \sigma_0^2 \sum_L^{-1}$, and that the least squares unbiased estimator of the covariance matrix of X is

$$\hat{\sum} \hat{X} = \hat{\sigma}_0^2 \left(A^T PA\right)^{-1} \tag{2.4}$$

Here we are dealing with linear mathematical models. Should the model be nonlinear, it must be linearized before the least squares method is applied.

3.0 Data reduction and adjustment

Gravity data reduction was performed using various steps (Aku et. al. 2002) [1]. The stations with absolute gravity determination provide the anchoring point (fixed points) of the network, while the relative measurements provide the ties between the points.

When the absolute and relative observations are made and assessed for accuracy, an adjustment can be carried out using a least-square adjustment technique. The adjustment results in the estimated values of gravity for all stations, together with their accuracy estimates. The adjustment procedure is practically identical with that of geodetic leveling (Poitevin and Ducarme, 1980) [5]. Gravity models attempt to describe in detail the variations in the gravity field. The importance of this effort is related to the idea of leveling, thus the gravity differences can be adjusted as a leveling network. This is because the summation of gravity differences around a closed loop theoretically goes to zero, and this condition can then be used as the basis for the adjustment. The following gravity network was observed.

	Δg_{AB}	Δg_{BA}	Δg_{BC}	Δg_{CD}	Δg_{DE}	Δg_{EF}	Δg_{FC}	Δg_{CF}	Δg_{FB}	Δg_{FA}	Δg_{FG}	Δg_{GF}
Gravity Difference (mGals)	0.143	-0.143	2..370	1.437	-0.897	-0.414	0.880	-0.779	-0..591	-0.635	1.206	-1.201
Time between stations (Hrs)	2	2	3	4	2	3	4	4	3	5	6	6

Table 1: Observed gravity network

Gravity at point A is supposed known and constant (980100.00 mGal) (Osazuwa, 1985) [3]. The weight of each observation is inversely proportional to the length of travel of the line.

Using the parametric method of least squares, the gravity of points B, C, D, E, F, and G and the adjusted observations $\hat{L}$ and the residuals $\hat{V}$ were computed.

From the given data, we have a number of observations n = 12 and the number of unknowns u = 6. Therefore we have 6 redundant observations (n – u) and 6 degrees of freedom. The mathematical model is $AX = L + V$. The 12 independent equations will be:

$$
\begin{aligned}
g_B &= g_A + 0.143 + V_1 \\
g_A &= g_B - 0.143 + V_2 \\
g_C &= g_B + 2.370 + V_3 \\
g_D &= g_C + 1.437 + V_4 \\
g_E &= g_D - 0.897 + V_5 \\
g_F &= g_E - 1.414 + V_6 \\
g_C &= g_F + 0.880 + V_7 \\
g_F &= g_C - 0.779 + V_8 \\
g_B &= g_F - 1.591 + V_9 \\
g_A &= g_F - 1.635 + V_{10} \\
g_G &= g_F + 1.206 + V_{11} \\
g_F &= g_G - 1.201 + V_{12}
\end{aligned}
\tag{3.1}
$$

where g_A has a constant value. Putting equation (3.1) in matrix form, we have

$$
A = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1
\end{vmatrix}, \quad
X = \begin{vmatrix} g_B \\ g_C \\ g_D \\ g_E \\ g_F \\ g_G \end{vmatrix}, \quad
L = \begin{vmatrix} 100.143 \\ -100.143 \\ 2.370 \\ 1.437 \\ -0.897 \\ -1.414 \\ 0.880 \\ -0.779 \\ -1.591 \\ -101.635 \\ 1.206 \\ -1.201 \end{vmatrix}
$$

The observations are considered uncorrelated with variances proportional to the corresponding travel time between stations. Thus the weight matrix is given as the inverse of the variance-covariance matrix of the observations, that is:

$$P = \text{diag}\,(1/2, 1/2, 1/3, 1/4, 1/4, 1/3, 1/4, 1/4, 1/3, 1/5, 1/6, 1/6)$$

The normal equations are $N\hat{X} = U$, yielding the solution $\hat{X} = N^{-1}U$

The *variance-covariance matrix* of an estimate $\hat{X}$ is obtained through the law of propagation of variance which states that if $z = f(x,y)$ and if x and y are not correlated, then

$$\sigma_z^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 .$$

Thus if $z = x - y$, $\sigma_2 = \sqrt{\sigma_x^2 + \sigma_y^2}$, where σ is standard deviation.

If X is a n-dimensional random variable with covariance matrix Σ_x, and a new n-dimensional random variable Y is computed through a linear relation $Y = AX + B$ with a constant coefficient matrix A and a constant vector B, then the covariance matrix Σ_y is given by $\Sigma_y = A.\Sigma_x.A^T$.

For non-linear relationships of the type $Y = f\,(X) + B$, A is replaced by the Jacobian matrix J containing the partial derivatives of f with respect to X.

Spreadsheet and a Fortran program was used to compute $\hat{X}$ and obtain the estimates for the gravity of points B, C, D, E, F, and G. We find:

$$\hat{X} = \begin{vmatrix} 100.133 \\ 102.515 \\ 103.971 \\ 103.083 \\ 101.684 \\ 102.887 \end{vmatrix} \text{ in mGals} \qquad \text{and} \qquad \hat{V} = A\hat{X} - L = \begin{vmatrix} -0.010 \\ 0.010 \\ 0.012 \\ 0.019 \\ 0.009 \\ 0.014 \\ -0.049 \\ -0.052 \\ 0.041 \\ -0.049 \\ -0.003 \\ -0.003 \end{vmatrix} \text{ in mGals}$$

Note: (980000 mGals to be added to $\hat{X}$). The adjusted observations are $\hat{L} = L + \hat{V}$

4.0 Conclusion

For precise gravity surveys, it is both logical and prudent to perform the weighted adjustment rather than the equal weight adjustment. Often results of physical measurements do not have the same precision since they have been made using different instruments by different persons, under different conditions. This fact is reflected in the least squares estimation process. The parametric method of least squares remains a reliable tool that greatly assists in overcoming obstacles that can stand on the way of producing a successful adjustment. A non-linear mathematical model must be linearized before the least squares method is applied.

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