Journal of the Nigerian Association of Mathematical Physics, Volume 9 (November 2005)

Simple mathematical models for housing allocation to a homeless population

Daniel Okuonghae
Department of Mathematics, University of Benin, Benin City, Nigeria.
e-mail:danny.okuonghae@corpus-christi.oxon.org

Abstract

We present simple mathematical models for modelling a homeless population and housing allocation. We look at a situation whereby the local authority makes temporary accommodation available for some of the homeless for a while and we examine how this affects the number of families homeless at any given time. We also take a look at priorities especially towards the homeless and see how this also affects the homeless in terms of housing allocation and examine steady states to see how all the group of families will fare after enough time has elapsed.

Keywords*:* Mathematical model, steady state.

pp 399 - 408

1.0 Introduction

In most countries, it is the responsibility of Local Housing Authorities or any other agency put in place by the government to place homeless people in government housing. They are also responsible for moving nonhomeless people from one government house to another. Since the demand for government houses exceeds the supply, this function is managed by operating waiting lists.

Some work has been done in the area of housing allocation and modelling of homeless populations. Byatt-Smith, J.G. et al [1] did some modelling in this area to see how changing priorities can affect waiting times and the size of a waiting list for council accommodation while Nikolopoulos et al [2] looked at housing allocation after a natural disaster must have occurred.

In this paper we derive and analyze models considering housing allocation to different categories of families including homeless families. We introduce the idea of temporary accommodation into the system and see the effect of this on those who are homeless.

The first model looks at a situation where we have the three general categories of families, viz: families that are homeless, those that live in government accommodation and those who live in private/general accommodation. We then introduce a fourth category: those that are taking from the homeless category and placed in temporary accommodation for a while, with some of such families are given some priority in getting government accommodation. The second model proposes an extension of the first one.

For both types of models, we derive non-linear systems of ordinary differential equations and analyze the stability of the systems. Next we find numerical solutions of the models and finally we consider possible extensions and improvements of the models.

2.0 The Mathematical model

In this section a simple model is derived consisting of a system of non-linear ordinary differential equations (ODEs). The population is divided into four different categories regarding the allocation state of the families. These are: (a) homeless families, (b) families in temporary accommodation, (c) families that live in government accommodation and (d) families that live in private/general accommodation. Modelling the flows between the families leads to a system of ordinary differential equations.

We also study some of the possible steady-states of the system and analyze the stability of the model for the positive equilibrium points that were found. We only restrict the analysis to states that shows the number of homeless families and/or those living in temporary accommodation tending towards zero.

2.1 Derivation of Model.

We consider four categories of the population:

(a) number of families that are homeless denoted by $H = H(t)$,

(b) number of families that are accommodated in a temporary facilities (families living in tents, prefabricated houses and so on provided by the government) denoted by $T = T(t)$,

(c) number of families that are living in government accommodation denoted by $\boldsymbol{C = C(t)}$ and

(d) number of families that are living in private/general accommodation denoted by $\boldsymbol{G = G(t)}$.

Figure 1 shows the schematic diagram showing the flow between these groups of families.

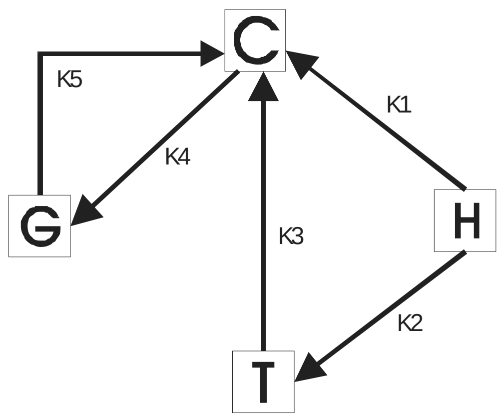

Figure 1: Schematic diagram showing the relationship between the four groups of families.

2.1.1 Assumptions

We make the following assumptions:

(i) Families that live in government or private/general accommodation will not become homeless. This is a reasonable assumption as most families tend to remain in their accommodations over a reasonable period of time.

(ii) We assume that the rate of change of number of homeless families accommodated in temporary accommodation will be jointly proportional to H and to the number of available temporary accommodation $T_o - T$, with constant of proportionality, k_2 and T_o being the number of temporary accommodation available in stock by the government. Also the number of homeless families being accommodated in government houses will be jointly proportional to H and to the availability of such houses, $C_o - C$, with constant of proportionality, k_1 and C_o being the number of houses available to the government.

(iii) We assume that the rate of change of number of families in temporary accommodation who are accommodated in government accommodation will be jointly proportional to T and the number of available government houses $C_o - C$ with constant of proportionality k_3. We also have k_5 being the rate at which families from the private/general group are accommodated in government houses. We also assume that the rate of families who leave government accommodation to private/general accommodation (perhaps due to improvement in circumstances) is a constant, k_4 which does not depend on the number of families that are already living in private/general accommodation.

(iv) We neglect birth and death rate.

(v) Number of families is large enough to be considered as continuum.
Hence the governing equations to the model are

$$\frac{dH}{dt} = -k_1 H(C_o - C) - k_2 H(T_o - T) \tag{2.1}$$

$$\frac{dT}{dt} = k_2 H(T_o - T) - k_3 T(C_o - C) \tag{2.2}$$

$$\frac{dC}{dt} = (k_1 H + k_3 T + k_5 G)(C_o - C) - k_4 C \tag{2.3}$$

$$\frac{dG}{dt} = k_4 C - k_5 G(C_o - C) \tag{2.4}$$

Assume further that we have a constant population $\boldsymbol{N}$ at any time, then

$$N = H(t) + T(t) + C(t) + G(t) \tag{2.5}$$

Using (2.5) we can eliminate (2.4) so that we have a system of three ordinary differential equations and one algebraic equation, for $t \geq 0$:

$$\frac{dH}{dt} = -k_1 H(C_o - C) - k_2 H(T_o - T) \tag{2.6}$$

$$\frac{dT}{dt} = k_2 H(T_o - T) - k_3 T(C_o - C) \tag{2.7}$$

$$\frac{dC}{dt} = (k_1 H + k_3 T + k_5 (N - C - T - H))(C_o - C) - k_4 C \tag{2.8}$$

$$G(t) = N - H(t) - C(t) - T(t) \tag{2.9}$$

Next we nondimensionalise. We scale quantities representing number of families H, T, C, G with P_o. Here we took P_o to be N. We then have that $H = x_1 N$, $T = x_2 N$, $C = x_3 N$, $G = x_4 N$ and as time, $\boldsymbol{t_o}$ i.e. $t = t_o \tau$ where $t_o = \frac{1}{k_1 N}$. Therefore the system of equations, for $\tau \geq 0$ becomes

$$\frac{dx_1}{d\tau} = -x_1(a - x_3) - \alpha x_1(b - x_2) \tag{2.10}$$

$$\frac{dx_2}{d\tau} = \alpha x_1(b - x_2) - \beta x_2(a - x_3) \tag{2.11}$$

$$\frac{dx_3}{d\tau} = (x_1 + \beta x_2 + \varepsilon(1 - x_1 - x_2 - x_3))(a - x_3) - \gamma x_3 \tag{2.12}$$

$$x_4(\tau) = 1 - x_1(\tau) - x_2(\tau) - x_3(\tau) \tag{2.13}$$

where $a = \frac{C_o}{N},\ b = \frac{T_o}{N},\ \alpha = \frac{k_2}{k_1},\ \beta = \frac{k_3}{k_1},\ \gamma = \frac{k_4}{k_1 N},\ \varepsilon = \frac{k_5}{k_1}$.

2.2 Stability of the System

To investigate the stability of the system, we find the equilibrium points which are solutions of the following set of equations:

$$-x_1(a - x_3) - \alpha x_1(b - x_2) = 0$$

$$\alpha x_1(b - x_2) - \beta x_2(a - x_3) = 0$$

$$x_1(a - x_3) + \beta x_2(a - x_3) + \varepsilon(a - x_3)(1 - x_1 - x_2 - x_3) = 0$$

(We dropped the γx_3 term in (2.12) as γ is taken to be very small).

Two interesting equilibrium points are $x_1 = 0, x_2 = 0, x_3 = 1$ and $x_1 = 0, x_2 = 0, x_3 = a$. Other equilibrium points can be obtained e.g. a state where we have no homeless people only after enough time has elapsed. The Jacobian $\boldsymbol{J}$ of the system is

$$\begin{pmatrix} -(a-x_3)-\alpha(b-x_2) & \alpha x_1 & x_1 \\ \alpha(b-x_2) & -\alpha x_1-\beta(a-x_3) & \beta x_2 \\ (a-x_3)(1+\varepsilon) & (a-x_3)(\beta+\varepsilon) & -x_1-\beta x_2-\varepsilon(1+a-x_1-x_2-2x_3) \end{pmatrix}$$

2.3 **Stability of the point** (0, 0, 1)

For this point, J has the eigenvalues $\lambda_1 = -\alpha b + (1-a), \lambda_2 = \beta(1-a), \lambda_3 = \varepsilon(1-a)$. The stability of this point depends crucially on the value of $\boldsymbol{a}$ i.e. on the availability of government accommodation.

If $a > 1$ then the three eigenvalues of the Jacobian of the system become negative and this point is asymptotically stable. This shows that if the number of government accommodation exceeds the population size, then it is possible for everyone to be in council accommodation after enough time has elapsed. Well this is a reasonable and obvious thing to expect. But if we are working with a system that has the number of government accommodation far less than the total population and the government is not making any move in building more houses, then we can not expect the outcome of this steady state. Some people will still remain homeless and some will still remain in temporary accommodation after enough time has elapsed.

If $a < 1$, we have that $\lambda_2 > 0, \lambda_3 > 0$. Hence the point (0, 0, 1) is not stable for this value of $\boldsymbol{a}$. In other words, while the number of government accommodation is less than the total number of families, it is not possible to have a situation where we will have everyone in council accommodation without having families in the homeless group or families still living in temporary accommodation and even without having people living in public accommodation after some time has elapsed. This is not realistic.

If $a = 1$, then $\lambda_2 = 0$, $\lambda_3 = 0$. Hence further analysis would be required to study this steady state.

2.4 **Stability of the point** (0, 0, a)

For this point, $\boldsymbol{J}$ has the eigenvalues $\lambda_1 = -\alpha b, \lambda_2 = 0, \lambda_3 = -\varepsilon(1-a)$. Again the stability of this point depends crucially on the value of a.

If $a > 1$, $\lambda_3 > 0$, hence the point (0, 0, a) will not be stable. Interestingly, we cannot actually consider $a > 1$ since in the four variable equations, this equilibrium point will be such that the number of those in private/general accommodation becomes negative which is not possible.

For $a < 1$, we have that $\lambda_3 < 0$. Already λ_1 is negative. Interestingly, $\lambda_2 = 0$ from the Jacobian matrix and not due to any condition imposed on it by the value of $\boldsymbol{a}$. The point is neutrally stable. We may still require further analysis to study the equilibrium point for these values of $\boldsymbol{a}$. We recall though that a critical point x^0 of a system of ODE's is said to be stable if as $t \to \infty$, the trajectory of the solution gets to the critical point or stay very close to it.

Figure 2 shows the numerical solution to the non-dimensional system of equations. This was solved using the Backward Euler scheme implemented with MATLAB. For some values of $\boldsymbol{k}$ (the unit is per households per year), we have that

$$a = 0.3612, b = 0.1547, \alpha = 0.3695, \beta = 0.1652, \varepsilon = 0.0024 .$$

The value of k_1 was taking to be larger than the others so as to give a higher priority to the homeless but low priority for others seeking government accommodation. The dotted curves shows the situation when we are not having families placed in temporary accommodation. The other curves show the effect of temporary accommodation. We see that the number of homeless families reduces rapidly. We observe the effect of the overall system on x_3, the nondimensional variable for families in government accommodation. Though x_2 is increasing, this starts to reduce as time goes on, although slowly. Those in private accommodation for most of the time remained unchanged.

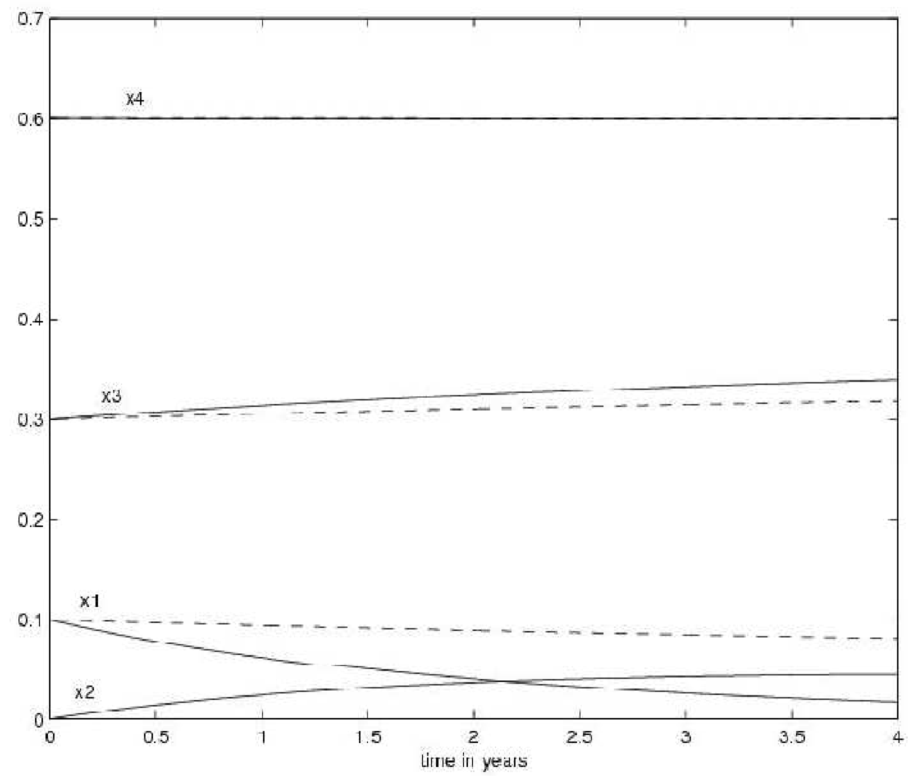

Figure 2: Numerical solution of the nondimensional model. The quantities x_i are plotted against time.

The other parameters also affected the rate of movements of the families. If α and β are increased, which signifies higher priorities to the homeless and those living in temporary accommodation in getting government houses, the number of such families reduces.

It is seen that temporary accommodation does affect the rate at which the homeless population is reduced since some in the homeless category will be leaving for the temporary category. We also note a significant rise in the number of those living in council accommodation. Even though some families remain in temporary accommodation for some time, this number also begins to reduce but very slowly.

3.0 Extensions to the Model.

The model we just considered can be improved upon. In the model we assumed that people do not become homeless. While this could be valid under a very short time period, this will not hold when the time period is long. Secondly, we assumed that the number of private/general accommodation is limitless which is also not a realistic assumption as the number of such houses is finite in any city. We shall incorporate these new assumptions into the existing model and see how the results look like.

3.1. **Derivation of Model**

We shall maintain the variables used and have the schematic of the new model to be what we have in figure 3 below.

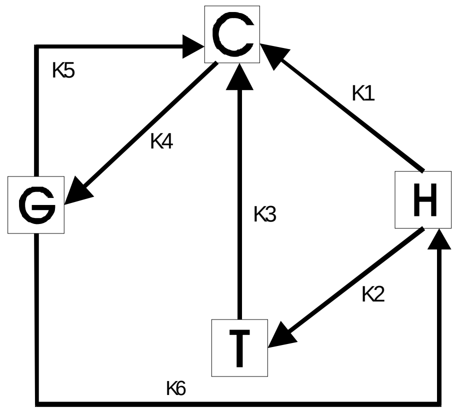

Figure 3: Schematic diagram of the extended model.

3.2 **Assumptions**

In addition to assumptions (ii-v) in (2.1.1), we make the following assumptions

(i) Families that live in government or private/general accommodation can become homeless. This is a reasonable assumption as families can become homeless say due to natural disasters, communal clashes, and unemployment and so forth.

(ii) We also assume that the rate of families who leave government accommodation to private/general accommodation (perhaps due to improvement in circumstances) is a constant, k_4 which depends on the number of available private/general accommodation $G_o - G$ where G_o is the total number of houses in the private/general sector. We have k_6 being the rate at which those in private accommodation joins the homeless group. This constant does not depend on the number of those already homeless.

The governing equations then becomes

$$\frac{dH}{dt} = -k_1 H(C_o - C) - k_2 H(T_o - T) + k_6 G \tag{3.1}$$

$$\frac{d\mathrm{T}}{d\mathrm{t}} = k_2 H(T_o - T) - k_3 T(C_o - C) \tag{3.2}$$

$$\frac{dC}{dt} = (k_1 H + k_3 T + k_5 G)(C_o - C) - k_4 C(G_o - G) \tag{3.3}$$

$$\frac{dG}{dt} = k_4 C(G_o - G) - k_5 G(C_o - C) - k_6 G \tag{3.4}$$

Again, we assume that we have a constant population $\boldsymbol{N}$ at any time, then

$$N = H(t) + T(t) + C(t) + G(t) \tag{3.5}$$

Using (3.5) we eliminate (3.4) so that we have a system of three ordinary differential equations and one algebraic equation, for $t \geq 0$:

$$\frac{dH}{dt} = -k_1 H(C_o - C) - k_2 H(T_o - T) + k_6(N - H - T - C) \tag{3.6}$$

$$\frac{d\mathrm{T}}{d\mathrm{t}} = k_2 H(T_o - T) - k_3 T(C_o - C) \tag{3.7}$$

$$\frac{dC}{dt} = (k_1H + k_3T + k_5(N - C - T - H))(C_o - C) - k_4C(G_o - N + H + T + C) \quad (3.8)$$

$$G(t) = N - H(t) - C(t) - T(t) \quad (3.9)$$

As usual we nondimensionalise the problem. Again we used the same scale we used in the first model. Therefore the system of equations, for $\tau \geq 0$ becomes

$$\frac{dx_1}{d\tau} = -x_1(a - x_3) - \alpha\, x_1(b - x_2) + \beta(d - x_1 - x_2 - x_3) \quad (3.10)$$

$$\frac{dx_2}{d\tau} = \alpha\, x_1(b - x_2) - \gamma x_2(a - x_3) \quad (3.11)$$

$$\frac{dx_3}{d\tau} = (x_1 + \gamma x_2 + \delta(d - x_1 - x_2 - x_3))(a - x_3) - \varepsilon x_3(c - d + x_1 + x_2 + x_3) \quad (3.12)$$

$$x_4[\tau] = d - x_1(\tau) - x_2(\tau) - x_3(\tau) \quad (3.13)$$

where

$$a = \frac{C_o}{P_o},\ b = \frac{T_o}{P_o},\ c = \frac{G_o}{P_o},\ \alpha = \frac{k_2}{k_1},\ \beta = \frac{k_6}{k_1P_o},\ \gamma = \frac{k_3}{k_1},\ \delta = \frac{k_5}{k_1},\ \varepsilon = \frac{k_4}{k_1},\ d = \frac{N}{P_o}.$$

3.3 Stability of the System

Let us take $\boldsymbol{P_o}$ to be $\boldsymbol{N}$ and making $\boldsymbol{k_5}$ very small, we see that β and δ are negligible, hence we set them to zero.

To investigate the stability of the system, we find the equilibrium points which are solutions of the following set of equations:

$$-x_1(a - x_3) - \alpha\, x_1(b - x_2) = 0$$

$$\alpha\, x_1(b - x_2) - \gamma x_2(a - x_3) = 0$$

$$x_1(a - x_3) + \gamma x_2(a - x_3) - \varepsilon x_3(c - 1 + x_1 + x_2 + x_3) = 0$$

The Jacobian $\boldsymbol{J}$ of the system is

$$\begin{pmatrix} -(a - x_3) - \alpha(b - x_2) & \alpha x_1 & x_1 \\ \alpha(b - x_2) & -\alpha x_1 - \gamma(a - x_3) & \gamma x_2 \\ a - x_3 - \varepsilon x_3 & \gamma(a - x_3) - \varepsilon x_3 & -x_1 - \gamma x_2 - \varepsilon(c - 1 + x_1 + x_2 + 2x_3) \end{pmatrix}$$

3.3.1 Investigating the Steady States

1. $(0, 0, 1 - c)$.

For this state, the eigenvalues of the Jacobian is

$$\lambda_1 = -(a + c - 1), \lambda_2 = -\gamma(a + c - 1), \lambda_3 = -\varepsilon(1 - c).$$

Clearly λ_1 and λ_2 will be less than zero if and only if $a + c > 1$ and $\lambda_3 < 0$ iff $c < 1$. In other words, if the combine number of government and private houses exceeds the total population then the state is stable and it is possible to bring the number of those homeless and living in temporary accommodation to zero after some times have elapsed. Worthy of note is that the number of private houses should not exceed the total population. This is an expected outcome.

2. $(0, 1 - (c + a), \boldsymbol{a})$.

Here, we wish to see if homelessness can be eradicated after enough time has elapsed. For this equilibrium point, the eigenvalues of the Jacobian is

$$\lambda_1 = -\alpha(b - 1 + c + a),$$

$$\lambda_2 = -\frac{\gamma}{2}(1 - c - a + \frac{\varepsilon\, a}{\gamma}) + \frac{1}{2}\sqrt{(\gamma - \gamma c - \gamma a + \varepsilon a)^2 - 4(\varepsilon a\gamma - \varepsilon ac\gamma - \varepsilon a^2\gamma)},$$

$$\lambda_3 = -\frac{\gamma}{2}(1 - c - a + \frac{\varepsilon\, a}{\gamma}) - \frac{1}{2}\sqrt{(\gamma - \gamma c - \gamma a + \varepsilon a)^2 - 4(\varepsilon a\gamma - \varepsilon ac\gamma - \varepsilon a^2\gamma)}$$

Obviously for this state to be stable, $\lambda_1, \lambda_2, \lambda_3$ must all be negative. In fact, $\lambda_1 < 0$ iff $b + a + c > 1$, $\lambda_2 < 0$ iff $c + a < 1$ and $\lambda_3 < 0$ iff $c + a < 1$. Anything short of this will render the steady state unstable. In other words, once the total numbers of houses in the city (temporary, government and private) are greater than the total population, then it is possible to drive the number of homeless families to zero, eradicating it totally. But this then implies that some families though few will remain in temporary accommodation for a long time. They will kind of remain in the 'coolers' for a long time.

3. $(-\gamma\frac{\alpha b + a}{\alpha}, \frac{\alpha b + a}{\alpha}, 0)$.

This steady state is not realistic as it implies that after enough time has elapsed, there will be no families in council accommodation. More so, one of the values of the steady state will be negative. This is not allowed and hence invalidates this steady state.

4. $(\frac{\gamma - a\gamma - c\gamma - b\alpha\gamma}{\gamma + \alpha - 1}, \frac{1 - a - c - b\alpha}{1 - \gamma - \alpha}, \frac{\alpha(b - b\gamma - 1 + c) + a(1 - \gamma)}{1 - \gamma - \alpha})$.

This steady state looks at a situation whereby we do not eliminate homelessness after enough time has elapsed and some families are still living in temporary accommodation, without having a decent housing of their own. However, analysis of this steady state could not produce a conclusive statement as at least one of the eigenvalues of the Jacobian from the use of this steady state was zero. Also it may not be easily know which of the values of the equilibrium point will be positive. This can only be achieved when severe restrictions are imposed on the other parameters other than the ones that govern the ratio of houses to the families a, b and c. This definitely may render the analysis futile and may not give a true picture of what happens in real life in different cities.

Figure 4 below gives a numerical solution of the nondimensional model. The equations were solved using the ODE15s solver in MATLAB. Here we took

$$a = 0.3612, b = 0.1547, c = 0.5747, \alpha = 0.3432, \gamma = 0.3826 \text{ and } \varepsilon = 0.0783.$$

We deliberately made k_2 slightly higher than k_1. Again we observe a little change in the number of families living in private accommodation, since we had a policy that allowed fewer families from the group getting government accommodation when there are more families in other categories in need of government accommodation. Obviously because of the restriction in the number of available houses (both government and private), the reduction in the homeless is rather gradual. The result obtained is clearly due to the choice of the parameters in the nondimensional model. With carefully chosen coefficients and initial conditions, interesting statements could be further inferred from the solutions obtained.

4.0 Summary and Conclusions.

We present two models for housing allocation especially to the homeless. In the first model we propose the idea of placing some homeless families in a temporary accommodation for a while. The second model did a little extension on the first to incorporate some more realistic situations. For both models, we found that the rate at which people go from being homeless to government houses really influence the number of homeless people. If more people become homeless (from the second model), then no significant effect is seen in reducing homelessness i.e. you have a rate of people becoming homeless than being housed. But if the rate at which people become homeless is smaller than the rate at which the homeless is housed (based on a policy that favours them), then there exist a significant effect on the number of homeless at any given time. Hence it is more important for the

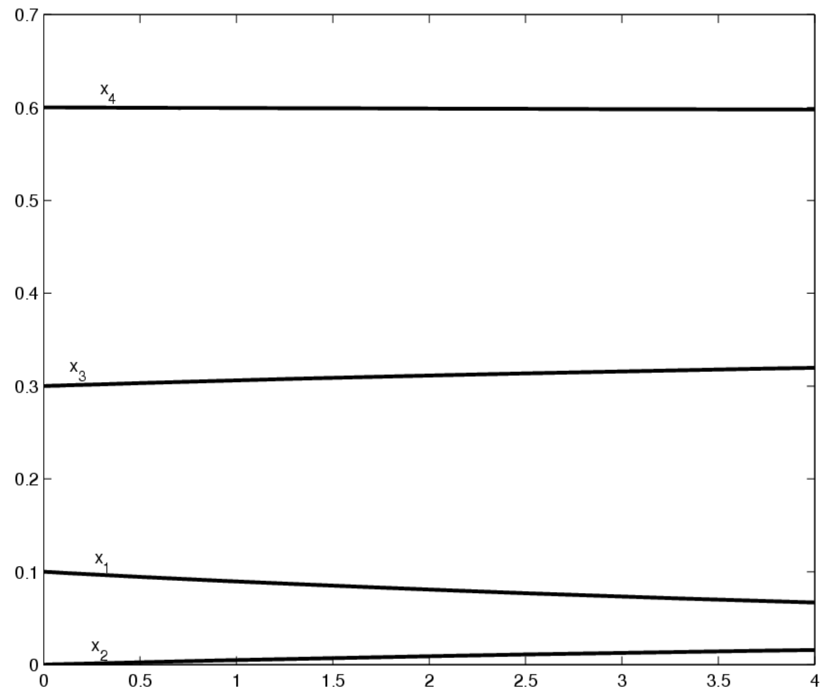

Figure 4: Numerical solution of the nondimensional model with the number of families plotted against time.

government to focus on societal issues that affect how people become homeless than on government housing policy.

We see a reduction in the number of homeless, with quite a few families spending a long time in temporary accommodation. The important variables in the models are the ratio of council houses to the entire population, a the ratio of temporary accommodation to the total population, b and the ratio of private/general accommodation to the entire population, c. This values which depends from city to city really had a role to play in determining how well the local authority can house the homeless in government accommodation with some increased priority.

Also an important variable in the models was k_1, the constant which determines the speed at which the homeless gets accommodated in government houses. If we reduced this constant, i.e. you give less priority to housing the homeless, it does little to affect others in the private/general accommodation category, but increases the number of families in temporary accommodation hereby forcing them to spend more time in such an accommodation (for the first model) or increase the number of homeless (for the second model).

Conclusively, we can say that though temporary accommodation reduces homelessness, most families will remain in such an accommodation for a while. Hence it is a case of taking people from the streets and placing them in a little better accommodation for some time. There was no significant effect such an accommodation had on those who get government accommodation. But if both the homeless and those already in temporary accommodation are given top priority for government houses, there respective number at any given time will be small. This priority will be determined by the choice of values for the parameters in the nondimensional models.

5.0 Future Work

We propose some future works, with some similar to those stated in Nikolopoulos et al [2]. We have assumed a constant population and also neglected death and birth rates. This not realistic over long term period as population density is dynamic and subject to changes over a long period of time. We also need to examine if our model holds true with data from city to city. Also we need to look at a situation where there exists a waiting list for government houses. This would have changed the overall dynamics of the problem.

Also we took the number of council houses in the first model and public/general houses in the second model to be a constant. This is not practicable especially over a long period of time. We could make this a function of time. For instance, we could say $C_o = C_e + \rho t,\ t_1 < t < t_2$ and zero elsewhere. We do the same for the number of houses in the private/general category. This modification would make the system of ordinary differential equations of the model nonautonomous and we should consider in such a case the time scales of the system more carefully as regards its stability.

Finally in order to be more precise as regards the determination of the coefficients and the initial conditions of the model we could consider these as random variables following some distribution (e.g. normal distribution), estimated by available data and come out with a system of stochastic differential equations.

References

[1] Byatt-Smith, J.G., Lacey, A.A., Parker, D.F., Simpson, D. and Smith, W.R. (2003). Mathematical Modelling of homeless population. *The Mathematical Scientists,* 28, 1-12.

[2] Nikolopoulos, C.V. and Tzanetis, D.E. (2003). A Model for housing allocation of a homeless population due to a natural disaster. *Nonlinear Analysis: Real World Applications,* vol 4, 4, 561-579.