Stationary population flow of a semi-open Markov Chain.

Bashir M. Yakasai
Department of Mathematical Sciences, Bayero University Kano, Nigeria.
e-mail: bmyakasai@yahoo.com

Abstract

In this paper we study the state vector of a semi-open Markov chain of a stochastic population flow. We consider stationary inflow of new members into the system and derive the limiting value of the state vector $X^{(n)} = \left(x_1^{(n)},\ldots,x_m^{(n)}\right) \in Z_+^{(m)}$ ***as*** $n \to \infty$***, when the system's capacity,*** η ***is known.***

Keywords: semi-open Markov chain, stationary inflow, state vector.

pp 395 - 398

1.0 Introduction

In the paper presented by Stadje [3] an open Markov chain was considered for stochastic population inflow model in which new individual members from outside are allowed to enter a system through any of the m ordered non overlapping strata $S_1, S_2, \ldots, S_m$ at discrete times n = 1,2,… . The paper focused on the number of individuals in each stratum at a discrete time n on the assumption that an individual independently stays in the stratum S_i with probability q_i, moves to the next immediate stratum S_{i+1} with probability p_i or leaves the system (stratum S_{n+1}) with probability r_i. He proved convergence of the joint distribution of group sizes for stationary process and also, derived limiting Laplace transform for the model. However, in this paper we present *semi-open* Markov chain that permits new members to join the system only through a particular specified stratum at discrete steps (times) n = 1,2,…. An individual in stratum i moves to stratum j with probability $p_{ij} \geq 0$ (i,j = 1,2,…,m). Our objective is to find stationary input that would be added to the system at each step of the process, so that the system's capacity is not exceeded up to the time n. And hence, determine its limiting value, as the numbers of steps grow very large.

2.0 Main problem.

2.1 Nomenclature.

P is m x m transition matrix.

$P^{(n)}$ is a matrix P raised to power n.

$Y^{(0)}$ is m-dimensional initial input vector with elements, $y_i^{(0)}; i = 1,2,\ldots,m$

$Y_1^{(n)}$ is nth step, m-dimensional input vector with nonzero element in first position and all other elements equal to zero.

$X^{(n)}$ is m-dimensional state vector for nth step process.

λ_i is the ith eigenvalue.

V_i is the left eigenvector associated with eigenvalue, λ_i.

U_i is the right eigenvector associated with eigenvalue, λ_i.

η is the system's capacity.

***Theorem* 2.1**

If P is an m x m stochastic transition matrix and $Y^{(0)} = \left(y_1^{(0)}, y_2^{(0)}, \ldots, y_m^{(0)}\right)$ *is the state vector containing the number of units in each of the m strata of the system at initial step of the process, then the expected number of units after n steps is given by*

$$X^{(n)} = \begin{cases} Y^{(0)}P & ; \quad n = 1 \\ Y^{(0)}P^{(n)} + \sum_{j=1}^{n-1} Y_1^{(n-j)}P^{(j)} & ; \quad n = 2,3,\ldots \end{cases} \qquad (2.1)$$

where $Y_1^{(n)} = \left(y_1^{(n)}, 0, \ldots, 0\right)$*;* $n \geq 1$ *is a new input admitted into the system at the nth step of the* process.

Proof

Consider the following recursive Markovian steps

Step 1: $X^{(1)} = Y^{(0)}P$

Step 2: $X^{(2)} = \left(Y^{(0)}P + Y_1^{(1)}\right)P$

Step 3: $X^{(3)} = \left[\left(Y^{(0)}P + Y_1^{(1)}\right)P + Y_1^{(2)}\right]P$

… … … … …

Step k: $X^{(k)} = \left\{\left[\left(Y^{(0)}P + Y_1^{(1)}\right)P + Y_1^{(2)}\right]P + \ldots + Y_1^{(k-1)}\right\}P$

Hence, the theorem follows immediately.

***Corollary* 2.2**

Suppose that new inputs are allowed to enter the system at constant rate at beginning of each step of the process. That is $Y_1^{(i)} = (y, 0, \ldots, 0) = Y$ *for all i (= 1,2…). Then, the state vector after n-steps is given by*

$$X^{(n)} = Y^{(0)}P^{(n)} + Y\sum_{j=1}^{n-1} P^{(j)} = X_a^{(n)} \quad ; \; n > 1 \qquad (2.2)$$

Next, suppose the system is not allowed to grow in size indefinitely, we then impose an absorbing barrier at state *m* of the chain. Now, consider equation (2.2), in which the input after each step is a fixed value, $Y = (y, 0, \ldots, 0)$. Then, if the system is restricted to accommodate a total number of η units, then we can determine the expected number of units that could be admitted into the system at each step, so that the system's capacity is sustained at the neighborhood of η. Therefore, from equation (2.2), we have

$$Y^{(0)}P^{(n)}\theta + \left(Y\sum_{j=1}^{n-1} P^{(j)}\right)\theta = \eta \qquad ; \text{n} > 1 \qquad (2.3)$$

where $\theta^1 = (1, 1, \ldots, 1, 0)$ is *m*-dimensional vector with all elements taking value 1 and the last element taking value zero, since the *m*th state of the chain is an absorbing barrier or removal state. Define

$$\sum_{j=1}^{n-1} P^{(j)} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mm} \end{pmatrix}$$

So that $$\left(Y\sum_{j=1}^{n-1}P^{(j)}\right)\theta = y\sum_{i=1}^{m-1}\alpha_{1i} \qquad (2.4)$$

From equations (2.3) and (2.4) we can show that

$$y = \frac{\eta - Y^{(0)}P^{(n)}\theta}{\sum_{i=1}^{m-1}\alpha_{1i}} \quad ; \quad n > 1 \qquad (2.5)$$

Hence, for $Y = \left(\frac{\eta - Y^{(0)}P^{(0)}\theta}{\sum_{i=1}^{m-1}\alpha_{1i}},\ 0,\ \ldots\ ,\ 0\right)$ we obtain the a^{th} element of $X_a^{(n)}$ as

$$x_a^{(n)} = \sum_{i=1}^{m} y_i^{(0)}p_{ia}^{(n)} + \frac{\alpha_{1a}}{\sum_{i=1}^{m-1}\alpha_{1i}}\left[\left(\eta - \sum_{i=1}^{m} y_i^{(0)}\right) + \sum_{i=1}^{m} y_i^{(0)}p_{im}^{(n)}\right] \text{ for } a = 1, 2, \ldots, m \qquad (2.6)$$

since $$Y^{(0)}P^{(n)} = \left(\sum_{i=1}^{m} y_i^{(0)}p_{i1}^{(n)}\ ,\ \ldots\ ,\ \sum_{i=1}^{m} y_i^{(0)}p_{im}^{(n)}\right)$$

and $$Y^{(0)}P^{(n)}\theta = \sum_{i=1}^{m} y_i^{(0)}\left(1 - p_{im}^{(n)}\right) \qquad (2.7)$$

where $p_{ql}^{(n)}$ is the *n*th power transition probability from state *q* to state *l* in *n*-steps of the chains.

3.0 Limiting value of $X_a^{(n)}$.

As *P* is an *m* x *m* stochastic transition matrix, then using Jordan canonical form (Cox and Miller [1]), we obtain the *n*th power of *P* as

$$P^{(n)} = M^{-1}D^nM \qquad (3.1)$$

where *M* is a nonsingular matrix whose rows are the left eigenvectors and the columns of M^{-1} are the right eigenvectors associated with eigenvalues λ_s ; s =1,2,…,m. D is a diagonal matrix with eigenvalue λ_s , *s* = 1, 2, … , *m* as its diagonal elements repeated according to multiplicity. Define $V_s = \left(v_1^{(s)},\ldots,v_m^{(s)}\right)$ and $U_s = \left(u_1^{(s)},\ldots,u_m^{(s)}\right)$ as left and right eigenvectors associated with the above eigenvalues and normalized so that $V_sU_s = 1$ (Iosifescu [2]). If the transition matrix P is primitive and $\lambda_m = 1$ we have the elements of $P^{(n)}$ given as

$$P_{rk}^{(n)} = \sum_{s=1}^{m-1} v_r^{(s)}u_k^{(s)}\lambda_s^n + v_r^{(m)}u_k^{(m)} \qquad (3.2)$$

where *r*,*k* = 1,2, …, *m* and $|\lambda_s| < 1$ for all $s \neq m$.

Also, from equation (3.1) the elements of the matrix $\sum_{j=1}^{n-1} P^{(j)}$ are obtained to be

$$\alpha_{rk} = \sum_{s=1}^{m-1} v_r^{(s)} u_k^{(s)} \left[\frac{\lambda_s \left(1-\lambda_s^{n-1}\right)}{1-\lambda_s} \right] + v_r^{(m)} u_k^{(m)} \qquad (3.3)$$

where $r,k = 1,2, \ldots, m$ and $|\lambda_s| < 1$ for all $s \neq m$.

Next, from equations (2.5) and (2.7) we can verify that

$$\lim_{n\to\infty} y \cong \frac{\eta - \sum_{i=1}^{m} y_i^{(0)} \left(1 - v_i^{(m)} u_m^{(m)}\right)}{\phi} = \tilde{y} \qquad (3.4)$$

where $$\phi = \lim_{n\to\infty} \sum_{i=1}^{m-1} \alpha_{1i} \cong \sum_{i=1}^{m-1} \sum_{s=1}^{m-1} v_1^{(s)} u_i^{(s)} \left(\frac{\lambda_s}{1-\lambda_s} \right) + v_1^{(m)} \sum_{i=1}^{m-1} u_i^{(m)}$$

provided that $$\eta + \sum_{i=1}^{m} y_i^{(0)} v_i^{(m)} u_m^{(m)} > \sum_{i=1}^{m} y_i^{(0)}$$

Also, from equation (2.6), (3.2) and (3.3) the *a*th position element of the state vector $X_a^{(n)}$ as $n \to \infty$ becomes

$$\lim_{n\to\infty} x_a^{(n)} \cong \sum_{i=1}^{m} y_i^{(0)} v_i^{(0)} u_a^{(0)} + \tilde{y} \left[\sum_{s=1}^{m-1} v_1^{(s)} u_r^{(s)} \left(\frac{\lambda_s}{1-\lambda_s} \right) + v_1^{(m)} u_a^{(m)} \right] \qquad (3.5)$$

for $a = 1, 2, \ldots, m$ and $|\lambda_s| < 1$ for all $s \neq m$.

4.0 Conclusion

We use semi-open Markov chain in this paper to mean that injection of the new inputs into the system is restricted to only one specified stratum. At each discrete step, new particles join the system from outside. We derive models for the state of the process after *n* number of steps (equation (2.6)) and the limiting behaviour of the system for a very long time duration (equation (3.5)). Both the models are based on the assumptions that the amount of new particles injected into the system after each step is constant and the system has limited capacity.

References

[1] Cox, D.R and Miller, H. D (1992). *The theory of stochastic process*. Chapman & Hall

[2] Iosifescu, M (1980). *Finite Markov process and their applications*. John Wiley & Sons.

[3] Stadje, W (1999). Stationarity of a stochastic population flow model. J. Appl. Prob. **36**, 291-294.