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Estimators of parameters of the exponential family distributions and maintenance problem.

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Abstract

In this paper we suggest a tabular structure for maintenance or reliability data. Also we propose a cost implication model of a preventive maintenance policy in which we included the average cost of downtime of the unit. Maximum likelihood and moment methods of estimations are used on some of the Exponential Family distributions; to make comparative study of the estimators and to demonstrate the application of the models presented with an arbitrary sampled data.

Keywords: *maximum likelihood, moments, exponential family, maintenance and failure-time.*

pp 389 - 394

1.0 Introduction

In maintenance or reliability problems the Exponential Family Distributions (EFD) plays a dominative role in determining hazard rate and or replacement date of units under prescribed policy. This is due to the fact that the Family of the exponential distribution have a common characteristic that fits any process undergoing steady increase or decrease in its modeling behavior. Distributions such as Exponential, Wiebull and Gamma attracted large number of contributors in the area of maintenance whenever it comes to the practical realization of their proposed models,. For instance, Phelps (1981) [12] used Weibull distribution of time to the failure of a reparable unit to make comparative study of the three different maintenance policies proposed by Park (1979) [11], Muth (1977) [10] and Barlow and Proschan (1965) [2]. Tango (1978) [13] used Erlang distribution in the paper in which he modified the used items replacement policy initially proposed by Bhat (1969). In the replacement policies proposed by Bashir (2000, 2001) [3] both Exponential and Weibull distributions where used. Exponential and or Wiebull distribution are also used by authors such as Christer (1984) [6], Lin (1988) [9] and many others in making realization of their models. In contrast, the Maxwell distribution is completely overlooked by the contributors. Another member of exponential family that is less attractive in maintenance problem is the normal density. Perhaps, this is due to its symmetric nature, since in practice it is not feasible to get a system or device whose failure-time distribution is normal. In this paper we focus on four members of the Exponential family distribution: Exponential, Wiebull, Gamma, and Maxwell distribution. We apply the two traditional methods of estimation, the maximum likelihood method (MLE) and Karl Pearson's moment method (MM) to estimate associated parameters of these distributions. We then apply the results to a sample data that is provided. Perhaps, the least square method can be of interest, however it is not considered in this communication. Friedman and Gertsbakh (1980) [8] discussed the existence and properties of maximum likelihood estimators for a minimum-type distribution function corresponding to a minimum of two independent random variables having Exponential and Wiebull distributions. We propose a tabular form of a maintenance-sampled data in section two to ease real application of some models of maintenance. In addition we modify the preventive maintenance policy presented by Cox (1982) [7], to incorporate downtime cost of the system.

2.0 Structure of a sample Data

Suppose we have *m* identically independent observed sample units. Each unit is likely to experience a number of defects or failures before a new identical unit replaces it. Also, assume that two or more defects could not arise at the time. However, if more than one defect arises simultaneously, we treat them under one repair. Define x_{ij} (i = 1, 2, …, m and j = 1, 2, …, n) to be the length of time the unit operates before failure or repair. Thus, we present our sample observed data as in Table 1, where $\overline{X}_{\cdot j} = \frac{1}{m}\sum_{i=1}^{m} x_{ij} = y_j$; j = 1, 2, … , n is the expected working-time of the unit before repair. The mean and variance of the working-time before the nth failure are given, respectively as

$$\bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j \text{ and } s_y^2 = \frac{\sum_{j=1}^{n} y_j^2 - n\bar{y}^2}{n-1} \tag{2.1}$$

3.0 Estimation of parameters

Consider the four members of the Exponential family: Exponential Distribution, Wiebull distribution, Erlang distribution and Maxwell distribution. We apply the methods of maximum likelihood and moment to determine the estimators of parameters of these distributions.

3.1 Exponential distribution

$$p.d.f.\ f(y) = \lambda \ell^{-\lambda y},\ y > 0 \tag{3.1}$$

$$\text{MLE: } \hat{\lambda}_1 = \frac{1}{\sqrt{\bar{y}}} \tag{3.2}$$

$$\text{MM: } \hat{\lambda}_2 = \frac{1}{\bar{y}} \tag{3.3}$$

Note: Subscript 1 refers to MLE and subscript 2 refers to MM estimator.

3.2 Weibull distribution.

$$\text{p.d.f. } f(y) = aby^{b-1}\ell^{-ayb}\ ;\ \text{y} > 0 \tag{3.4}$$

$$\text{MLE: } \hat{a}_1 = \frac{m}{\sum y_i^{\hat{b}}} \quad \text{and} \quad \hat{b}_1 = \left\{\left(1 + \log\prod_{i=1}^{m} y_i\right)\prod_{i=1}^{m} y_i\right\} \tag{3.5}$$

MM: In this case we find $\hat{b}_2$ such that

$$\frac{\hat{b}_2 \Gamma\left(\frac{2}{\hat{b}_2}\right)}{\left[\Gamma\left(\frac{1}{\hat{b}_2}\right)\right]^2} = 1 + \frac{s_y^2}{\bar{y}^2} \tag{3.6}$$

And hence we obtain $\hat{a}_2$ such that

$$\left(\frac{1}{\hat{a}_2}\right)^{\frac{1}{\hat{b}_2}} = \frac{\bar{y}\hat{b}_2}{\Gamma\left(\frac{1}{\hat{b}_2}\right)} \tag{3.7}$$

3.3 **Gamma distribution**

$$\text{p.d.f.: } f(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} \ell^{-\frac{y}{\beta}}; \; y > 0 \quad (3.8)$$

$$\text{MLE: } \hat{\alpha}_1 = \frac{\bar{y}}{\hat{\beta}_1} \quad \text{and} \quad \hat{\beta}_1 = \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}} \quad (3.9)$$

$$\text{MM: } \hat{\alpha}_2 = \frac{\bar{y}}{\hat{\beta}_2} \quad \text{and} \quad \hat{\beta}_2 = \frac{s_y^2}{\bar{y}} \quad (3.10)$$

3.4 **Maxwell distribution**

$$\text{p.d.f: } f(y) = \sqrt{\frac{2}{\pi}} \theta^{\frac{3}{2}} y^2 \ell^{-\frac{\theta y^2}{2}} \quad ; \quad y > 0 \quad (3.11)$$

$$\text{MLE: } \hat{\theta}_1 = \frac{3}{\bar{y}^2 + \left[\frac{m-1}{m}\right] s_y^2} \quad (3.12)$$

$$\text{MM: } \hat{\theta}_2 = \frac{8}{\pi \bar{y}^2} \quad (3.13)$$

4.0 Empirical study

Suppose we have a statistical record of an automobile engine failure-times history, as given in Table 2, where $x_{ij}; (i = 1,2,\cdots,m(=5) \;\; \textit{and} \;\; j = 1,2,\cdots,n(=3))$ is the length of time the engine operates before jth failure occurs. Next we have that $n = 3$; $\sum_{j=1}^{n} y_j = 22.78$; $\sum_{j=1}^{n} y_j^2 = 294.02$; $\bar{y} = 7.593$ and $s_y^2 = 60.60$. From table 3, by simple comparison of the sample mean with the mean of the distributions, gamma distribution fits our observed data more than any other competing distribution, since under MLE and MM it provides better results. Also, under MM the estimators of the parameters in all the cases performed well and yielded better result than MLE. In such situation, we suggest that the distribution which gives parametric mean closer to the sample mean and having least value of the standard error is recommended for the observed data. Therefore, the Gamma distribution under MLE is the recommended distribution as for our observed data.

5.0 Maintenance policy

In this section we consider a simple replacement strategy, that is

A unit is replaced at planned periods kT, where k=1,2, … and if failure occurs before the preventive replacement period, repair is conducted.

Next define c_p; c_s ($> c_p$) and c_d to present the average costs for planned replacement, repair and per unit downtime cost, respectively. Hence, without lost of focus on point process involved in this case, we have the expected cost for adopting the above policy (strategy) as

$$C(T)=\frac{c_p+c_sH(T)+c_d\left[T-\int_0^T\overline{F}(t)dt\right]}{T}, \qquad T>0 \qquad (5.1)$$

where $H(T)=F(T)+\int_0^T H(T-t)dF(t)$ is renewal function.

And $\int_0^T\overline{F}(t)dt=\int_0^T[1-F(t)]dt$ is a cumulative working-time of the system in the interval (0,T)

Our objective now is to seek for the optimum value of $T = T^*$ that minimize $C(T)$. Thus, by taking partial derivative of equation (15) with respect to T and equate to zero, we obtain that

$$c_s[Th(T)-H(T)]+c_d\left[TF(T)-\int_0^T F(t)dt\right]=c_p \qquad (5.2)$$

Where $h(T)=\frac{dH(T)}{dT}$ is a renewal density. To solve equation (5.2) we suggest numerical approximation or analytical method. To make a realization of (5.1) and (5.2) we consider for simplicity sake an Erlangian (Gamma) distribution with one stage and a Maxwell distribution. From the p.d.f of the Erlang

we get $$F(T)=1-\sum_{r=0}^{a-1}\frac{\left(\frac{T}{\beta}\right)^r}{r!}\ell^{-\frac{T}{\beta}} \qquad (5.3)$$

which is an in complete gamma function, Tango (1978). Thus for a =1

$$\int_0^T F(t)dt=T-\beta\left(1-\ell^{-\frac{T}{\beta}}\right) \qquad (5.4)$$

And since H(T) is the expected number of failure in the interval (0, T] then according to Akimaru and Kawashima (1993) [1], $H(T)=T/E(y)=T/\alpha\beta$ and hence $h(T)=1/\alpha\beta$.. Then, equation (5.2) becomes

$$(\beta+T)\ell^{-\frac{T}{\beta}}=\beta-\frac{c_p}{c_d} \qquad (5.5)$$

Thus, the optimum value of T=T* is determined so that equation (19) is satisfied.

$$C(T^*)=\frac{c_p+c_sH(T^*)+c_d\left\{T^*-\beta\left(1-\ell^{-\frac{T^*}{\beta}}\right)\right\}}{T^*}\ ;\ T^*>0 \qquad (5.6)$$

Optimality existence.

For the optimal value to exist, we observe that the left hand side of equation (5.5) is nonnegative for all $T\geq 0$, and hence, by implication it means that the equality exist only if $\beta\geq\frac{c_p}{c_d}$. Consider our empirical results and suppose c_p = 1, c_s = 1.25 and c_d= 11.75. Then from equation (5.5), $T=T_e^*\cong 2.95$ and $C(2.95)$ = 0.911 under MLE and $T=T_e^*\cong 3.47$ with $C(3.47)$ = 3.354 under MM. These results confirm that the estimation method that yields the least standard error is better. provided the methods give the same means.

The Maxwell distribution is given by

$$F(T)=\left(\sqrt{\frac{2}{\pi}}\right)\frac{\theta^{\frac{5}{2}}}{2\theta-1}T^{3}\ell^{-\frac{\theta T^{2}}{2}} \tag{5.7}$$

The expected value of this mass function is $\sqrt{\frac{8}{\theta\pi}}$ so that $H(T)=\frac{T}{2}\sqrt{\frac{\theta\pi}{2}}$ and hence $h(T)=\frac{1}{2}\sqrt{\frac{\theta\pi}{2}}$.

Therefore, from equation (5.2) we find optimum value of $T=T_M^{*}$ such that

$$\left[2(\theta T+1)^{2}+1\right]\ell^{-\frac{\theta T^{2}}{2}}=\frac{c_p(2\theta-1)}{c_d}\sqrt{\frac{2\pi}{\theta}}+3 \tag{5.8}$$

Optimality existence.

The left hand side of equation (5.8) is nonnegative for value of $T\geq 0,$ hence the right hand side also is nonnegative only if

$$\frac{1-2\theta}{\sqrt{\theta}}\leq\frac{3}{\sqrt{2\pi}}\left(\frac{c_d}{c_p}\right). \tag{5.9}$$

If the optimality condition is satisfied then we determine T = T* such that equation (5.8) holds. And hence the expected cost of the policy is obtain as

$$C(T^{*})=\frac{c_p+c_s\left(\frac{T^{*}}{2}\sqrt{\frac{\theta\pi}{2}}\right)+c_d\left[\sqrt{\frac{2\theta}{\pi}}\left(\frac{3\theta-(T^{*}-3\theta)\ell^{-\frac{1}{2}T^{*2}}}{2\theta-1}\right)\right]}{T^{*}};\ T^{*}>0. \tag{5.10}$$

Consider our numerical results for the Maxwell distribution (under MM) we obtain from equation (22) that T^{*} = 5.57 is the more suitable value for that equation. And hence from equation (5.10), C(5,57) = 1.357.

6.0 Conclusion

In this presentation we propose a maintenance statistical table for a coherent system failure-times history; and we modify the traditional planned replacement policy by taking into consideration the downtime cost of the system. Also a survey of the exponential family distributions is conducted to determine which of the distributions fits our observed data. The results show that the Erlang distribution with maximum likelihood estimators is better with the expected cost of the maintenance policy 0.911 and preventive replacement period 2.95.

Table 1: Observed working status of units after repair.

Units (*i*)	**Number of repairs (failures) (*j*)**				**Total**
	1	2	…	*n*	
1	X_{11}	X_{12}	…	X_{1n}	$X_{1\cdot}$
2	X_{21}	X_{22}	…	X_{2n}	$X_{2\cdot}$
:	:	:	:	:	:
M	X_{m1}	X_{m2}	…	X_{mn}	$X_{m\cdot}$
Total	$X_{\cdot 1}$	$X_{\cdot 2}$	…	$X_{\cdot n}$	

Note: *n* is the number of repairs and m is the number of units (machines).

Table 2: Machine failure time history.

Machines	Number of failures or repairs (*j*)		
	1	2	3
1	17.1	4.8	2.5
2	15.3	3.0	2.5
3	18.5	4.5	2,7
4	15.0	4.9	1,9
5	16.7	2.8	1.7
Total	82.62	20.00	11.30
$\bar{y}_j$	16.52	4.00	2.26

Table 3: Estimates of the parameters with their means and standard errors.

P.d.f	**Parameter**	**Mean**	**S.E.**
Exponential	$\hat{\lambda}_1 = 0.363$ (MLE)	2.755	*
	$\hat{\lambda}_2 = 0.132$ (MM)	7.576	7.576
Gamma	$\hat{\alpha}_1 = 1{,}431$; $\hat{\beta}_1 = 5.306$ (MLE)	7.593	6.347
	$\hat{\alpha}_2 = 0.951$; $\hat{\beta}_2 = 7.984$ (MM)	7.593	7.786
Weibull	$\hat{b}_1 = 21.772$; $\hat{a}_1 = 9.0832 \times 10^{-27}$ (MLE)	15.285	*
	$\hat{b}_2 = 0.570$; $\hat{a}_2 = 0.414$ (MM)	7.584	14.226
Maxwell	$\hat{\theta}_1 = 0.028$ (MLE)	9.537	*
	$\hat{\theta}_2 = 0.044$ (MM)	7.608	3.211

* The SE is not obtained because the parametric mean is not encouraging.

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