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# Performances of estimators of linear model with auto-correlated error terms when the independent variable is normal

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**Abstract**

***A Monte Carlo Study of the small sampling properties of five estimators of a linear model with Autocorrelated error terms is discussed. The independent variable was specified as standard normal data. The estimators of the slop coefficients $\hat{\beta}_2$ with the help of Ordinary Least Squares (OLS), increased with increased autocorrelation especially when T is small. On the other hand, the same slope coefficients $\hat{\beta}_2$, under Generalized Least Squares (GLS) decreased with increased autocorrelation when the sample size T is small.***

**Keywords**: Autocorrelation, Monte Carlo, Estimator, Standard Normal.

**pp 379 – 384**

## 1.0 Introduction

Approaches to dealing with estimation in Autocorrelated linear models include overall Maximum likelihood estimation, Ordinary Least Squares and transformation of variables. When the Autocorrelation coefficients are known, usually, the estimation poses no major problems as the underlying variables can be transformed to overcome this problem. Different forms of transformation techniques have been proposed by different authors. Several authors have different methods of estimating the Autocorrelated parameters in situations where the variables are unknown. These error estimates are used as weights in estimating $\beta$, the regression coefficient. For this study, we used five estimators namely the Cochrane and Orcutt estimator (COC), Hildreth and LU (HILU), the maximum likelihood Grid (MLGRID), the maximum likelihood (ML) and the Ordinary Least Squares (OLS). All the other estimators apart from the Ordinary Least Squares (OLS) estimator are called Generalized Least Squares (GLS) estimators.

Asymptotically, each of these estimators are equivalent with identical asymptotic properties. But in finite samples, such as in this work, Park and Mitchell (1980) [7] have argued that the estimation methods that use the P transformation matrix are generally more efficient than the estimators that use P transformation matrix. Kramer (1980) [4] in his paper stated that the efficiency of these estimators depends much on the specification of the independent variable used in the experiment and went ahead to show that some of the estimators proposed in the literature for example, Cochrane-Orcutt, has lower efficiency than OLS especially when the independent variable has some-trend.

Iyaniwura and Nwabueze (2004) [2] in their work, estimated the autocorrelated error model with Gross National Product (GNP) data as the independent variable and found out that the estimators COC and HILU performed worse than OLS while MLGRID and ML performed better than OLS. Also, Nwabueze (2005) [6] in a work on Autocorrelated error linear model discovered that when the independent variable is autoregressive, the variance of the slope coefficient of the GLS methods increase very slowly while the variance of the slope coefficient of the OLS increases very sharply with increasing value of $\rho$ and was always larger than the variance of the GLS estimators.

Some authors like Macshiro (1976) [5], Beach and Mackinnon (1978) [1] and Park and Mitchell (1980) [7] have investigated the efficiency of some of these estimators over OLS. This study broadens and deepens our understanding of the finite sampling properties of some of the estimators of linear models with auto-correlated error terms prevalent in the literature and examines the effects of standard normal data as the independent variable on the performances of these estimators. Also, this work investigates the effects of increased sample size and the degree of the Autocorrelation coefficient of the error terms on the performances of these estimators.

## 2.0 Estimation methods

Consider the common linear econometric model.

$$\left.\begin{array}{l} Y_t = X\beta + U_t;\ U_t = \rho U_t^{-1} + \varepsilon_t,\ t = 1, \cdots T.|\rho| \prec 1 \\ E(\varepsilon_t) = 0,\ E(\varepsilon\varepsilon^1) = \sigma^2\varepsilon 1,\ E(U) = 0 \ and\ E(UU^1) = \sigma^2\,\Omega \end{array}\right\} \qquad (2.1)$$

On multiplying the model 2.1 by some non-singular transformation matrix P of order TXT, we obtain.

$$PY = PX\beta + PU \qquad (2.2)$$

But the variance matrix of the disturbance in equation (2.2) is E $(PUU^1P) = \sigma^2 P\Omega P^1$, since E(PU) = 0. Therefore, if it were possible to specify $P$ such that $P \cap P^1 = 1$, then resulting OLS estimates of the transformed variable PY in equation (2.2) have all the optimal properties of OLS and could be validly subjected to the usual inference procedures (Johnson and Dinardo, 1997) [3]. Applying OLS to equation (2.2) results in minimizing the quadratic form

$$U^1\Omega^1 U = (Y - X\beta)^1\Omega^1 (Y - X\beta) \qquad (2.3)$$

with optimal solution as

$$\frac{\delta}{\delta\beta}\left(U^1\Omega U\right) = \left(X^1\overline{\Omega}^1 X\right)\beta - X^1\overline{\Omega}^1 Y = 0 \qquad (2.4)$$

which gives

$$\hat{\beta}(GLS) = (X^1\Omega^{-1}X)^{-1} X^1\Omega^{-1}Y \qquad (2.5)$$

The variance covariance matrix is given as var$\left(\hat{\beta}\right) = \sigma^2(X^1\Omega^{-1}X)^{-1}$. This estimator $\hat{\beta}$(GLS) is known as the Aitken or Generalized Least Squares (GLS) estimator. If we assume normality for the error terms, the likelihood function is given by:

$$L\left(\overline{\beta}, \frac{\sigma^2}{y}\right) = \left(2\pi\sigma^2\right)^{-\frac{T}{2}} |\Omega|^{-\frac{1}{2}} \exp\left\{\frac{(y - X\beta)^1\Omega^{-1}(y - X\beta)}{2\sigma^2}\right\} \qquad (2.6)$$

where $|\Omega|$ is the determinant of $\Omega$. Optimizing the likelihood function of equation (2.6) with respect to $\beta$, means maximizing the weighted sum of squares and we obtain

$$\hat{\beta}(GML) = (X^1\Omega^{-1}X)^{-1}X^1\Omega^{-1}Y \qquad (2.7)$$

In obtaining $\hat{\beta}$(OLS) and $\hat{\beta}$(GLS), we assume that $\Omega$ is known, when $\Omega$ is not known, we resort to estimating $\Omega$ by $\hat{\Omega}$ in which case, we obtain an estimated Generalized Least Square (EGLS) or estimated generalized maximum likelihood (EGLM) estimator and therefore, $\hat{\beta}$(GLS) = $(X^1\hat{\Omega}^{-1}X)^{-1}X^1\Omega^{-1}Y$. For this model in equation (2.1), the TXT covariance matrix of the error vector is

$$E\left(UU^1\right) = \sigma_u^2 V = \sigma_\varepsilon^2 \left\{\begin{array}{ccccc} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho^{T-1} & \cdots & \cdots & \cdots & 1 \end{array}\right\} \qquad (2.8)$$

where $\sigma_u^2 = \frac{\sigma_\varepsilon^2}{\left(1-\rho^2\right)}$. To search for a suitable transformation matrix P*, we consider the following matrix P* of order (T - !)XT defined by

$$P* = \begin{Bmatrix} -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\rho & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -\rho & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \rho^{T-1} & \cdots & \cdots & \cdots & 1 & \end{Bmatrix} \qquad (2.9)$$

where

$$E\left(UU^1\right) = \sigma_u^2 V = \sigma_\varepsilon^2 \begin{Bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho^2 & 1+\rho^2 & \rho & \cdots & 0 & 0 \\ 0-\rho & 1 & \rho^2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{Bmatrix}_{(T-1)XT} \qquad (2.10)$$

P*$^1$P* gives (1 - $\rho^2$) $\Omega^{-1}$ with -$\rho^2$ instead of 1 as the first element. Next, we consider another TXT transformation matrix P obtained by adding a new first row with 1 - $\rho^2$ in the first position and zero elsewhere.

$$P = \begin{Bmatrix} 1-\rho^2 & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{Bmatrix}_{TXT} \qquad (2.11)$$

$P^1P = (1-\rho^2)\,\Omega^{-1}$

P* and P differ only in the treatment of the first observation. P* is much easier to use provided one is prepared to put up with its treatment of the first observation. It has been shown that when *T* is large, the difference is negligible but in small samples such as in this work, the difference is significant. Such transformations give rise to different methods of estimation. These methods are broadly classified into those that use P* transformation matrix such as Cochrane-Orcutt (COC) and Hildreth and LU(HILU) estimators and those that use P transformation matrix such as Maximum Likelihood (ML) estimators of Beach and Mackinon (1978) [1] and Maximum Likelihood method Grid (MLGRID). The values of the autocorrelation coefficient of the error term rho (ρ) used were 0.4, 0.8 and 0.9. These values were chosen in order to determine the effect of autocorrelation on the performance of the estimators

## 3.0 **Method of simulation**

In econometrics, while asymptotic properties of estimators obtained by using various econometric techniques are deduced from postulates, an approach that is often described as analytical, small sample properties of such estimators have always been studied from simulated data known as Monte Carlo studies. The use of this approach is due to the fact that real life observations on economic variables are in most cases plagued by one or all of the problems of non-spherical disturbances and measurement errors. This work uses Monte Carlo Approach. The parameter values of $\beta_1$ and $\beta_2$ in the model (2.1) were fixed at (1,1). To generate multivariate normal vectors to be used for this study, the autocorrelated error term $U_t = \rho U_{t-1} + \gamma_t$ was first generated.

After generating the autoregressive error terms, a set of data that is standard normal was generated as the independent variable. Thereafter, the multivariate normal dependent vectors Y were computed using equation (2.1).

The generation of the error terms, the independent variable and the computations of the dependent variable are made using a package for econometric studies called time series processor (TSP). The simulation experiment was replicated 50 times. The sample sizes were varied from 20 to 40 to 60 in order to test the effect of sample size on the performance of the estimators. Since this study is investigating the performance of the estimators when the error term is autocorrelated, three different estimates of the error term $\rho$ were used for the study namely when $\rho$ = 0.4, 0.8 and 0.9. Thereafter, different estimation methods were applied to the data using the AR (1) function of the TSP Software package on an IBM Computer at the center for econometric and Allied research (CEAR), University of Ibadan. The performance of the simulated values from the original data series based on the estimators were then assessed by simulation statistics. The simulation statistics used in assessing the performances of the estimators in this study are: bias, sum of bias of both the intercept and slope coefficients (SBIAS), the variance, sum of variance of both the intercept and slope coefficient (SVARS), the root mean square error (SRMSE), and the sum of root mean square error (SEMSE) of both the intercept and slope coefficients.

## 4.0 Results and discussions

Tables 1-3 summarize the relative performances of the five estimators using the criteria, Bias, sum of bias (SBIAS), variance (VAR), sum of variance (SVAR), root mean square error (RMSE) and sum of root mean square error (SRMSE). In the first column are the three values of the autocorrelation coefficient $\rho$ used for our study. The performances of these estimators were also compared using different sample sizes so the first row of the tables shows the different sample sizes used for the experiment. The results of our experiment reveal that of the basis of BIAS property, the GLS method compares very well with the OLS and with one another when the size of the sample and degree of autocorrelation are small.

In this case, the OLS and the GLS methods underestimate the slope and the intercept coefficients (i. e when $\rho$ and T are both small). These biases are observed to decrease with increased sample size. As $\rho$ increases and T is small, the biases for COC increase. This shows that the effect of degree of autocorrelation on the comparative performance of the estimators is of some significance when T is small. The five methods yield the SBIAS of 0.142415, 0.395290, 0.025170, 0.147572 and 0.151300 for T =20 and $\rho$ = 0.9 for OLS, COC, HILU, MLGRID, and ML respectively (see Table 1).

## 5.0 Conclusion

An evaluation of the estimators using the least variance criterion reveals that:

1. The estimates of the slope coefficient $\beta_2$ (OLS) increases with increased autocorrelation especially when T is small.
2. The estimates of the slope coefficient $\beta_2$ (GLS) decreases with increased autocorrelation when T is small.

The reason for this difference between OLS and GLS estimators in the estimator of the slope coefficient $\beta_2$ could be due to the fact that the Generalized Least Squares (GLS) estimators correct the autocorrelation of the error term unlike the Ordinary Least Square (OLS) estimator which assume that the error terms are uncorrelated.

3. In estimating this model when T is small and $\rho$ is large, the estimators in order of preference are: HILU, ML, MLGRID, OLS and COC ($\rho$ = 0.9, T = 20) and
4. In estimating this model when T is large, the order of preference is COC, HILU, ML, MLGRID and OLS.

The same conclusions are reached when the estimators are evaluated using the root mean square error criterion.

**Table 1:** The Use of Bias to Compare the Estimators

| | | T = 20 | | | T = 40 | | | T = 60 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| ρ | ESTIMATORS | $\beta_1$ | $\beta_2$ | **SBIAS** | $\beta_1$ | $\beta_2$ | **SBIAS** | $\beta_1$ | $\beta_2$ | **SBIAS** |
| | OLS | -0.029770 | -0.055928 | 0.085698 | 0.000689 | -0.000385 | 0.001074 | -0.000758 | -0.035316 | 0.036074 |
| | COC | -0.024271 | -0.026089 | 0.050360 | 0.003754 | -0.001620 | 0.005374 | -0.001260 | -0.024480 | 0.025740 |
| 0.4 | HILU | -0.026970 | -0.024769 | 0.051739 | 0.004821 | -0.001167 | 0.005988 | -0.001094 | -0.020610 | 0.021704 |
| | MLGRID | -0.025078 | -0.025375 | 0.050453 | 0.002090 | -0.000500 | 0.002590 | 0.001363 | -0.021293 | 0.022656 |
| | ML | -0.026880 | -0.028550 | 0.055430 | 0.002300 | -0.015100 | 0.017400 | 0.001445 | -0.021960 | 0.023406 |
| | OLS | 0.032157 | -0.023052 | 0.055209 | 0.013230 | -0.025790 | 0.039020 | 0.002391 | -0.035180 | 0.037571 |
| | COC | -0.158237 | 0.007419 | 0.165656 | 0.000077 | 0.004206 | 0.004283 | 0.002122 | -0.008810 | 0.010932 |
| 0.8 | HILU | -0.068225 | -0.005345 | 0.073570 | 0.002958 | -0.000119 | 0.003077 | 0.005173 | -0.005680 | 0.006197 |
| | MLGRID | 0.045290 | 0.007323 | 0.052613 | 0.001573 | 0.015450 | 0.017023 | 0.009774 | -0.008403 | 0.018177 |
| | ML | 0.044990 | 0.006860 | 0.051850 | 0.003790 | 0.000770 | 0.010590 | 0.021790 | -0.037500 | 0.017802 |
| | OLS | 0.100235 | -0.042180 | 0.142415 | 0.026091 | -0.029320 | 0.053411 | 0.021790 | -0.037500 | 0.059290 |
| | COC | 0.392290 | 0.003000 | 0.395290 | 0.025300 | 0.000542 | 0.025842 | 0.015147 | 0.006220 | 0.021367 |
| 0.9 | HILU | -0.021570 | 0.003600 | 0.025170 | 0.031960 | 0.000730 | 0.032690 | 0.019210 | -0.006089 | 0.025299 |
| | MLGRID | 0.133016 | 0.014556 | 0.147572 | 0.013668 | -0.000270 | 0.013938 | 0.028649 | 0.004554 | 0.033203 |
| | ML | 0.141090 | 0.010210 | 0.151300 | 0.014190 | -0.014900 | 0.000710 | 0.014900 | -0.004691 | 0.034300 |

**Table 2:** The Use of Variance to Compare the Estimators

| | | T = 20 | | | T = 40 | | | T = 60 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| ρ | ESTIMATORS | $\beta_1$ | $\beta_2$ | **SVAR** | $\beta_1$ | $\beta_2$ | **SVAR** | $\beta_1$ | $\beta_2$ | **SVAR** |
| | OLS | 0.001147 | 0.054620 | 0.055767 | 0.000589 | 0.029980 | 0.030539 | 0.000250 | 0.025890 | 0.026140 |
| | COC | 0.035619 | 0.040400 | 0.076019 | 0.002426 | 0.024204 | 0.026630 | 0.000870 | 0.019850 | 0.020717 |
| 0.4 | HILU | 0.051900 | 0.039980 | 0.091880 | 0.002581 | 0.002575 | 0.026156 | 0.000858 | 0.022050 | 0.022908 |
| | MLGRID | 0.003440 | 0.040771 | 0.044211 | 0.006490 | 0.023810 | 0.04459 | 0.000404 | 0.019816 | 0.020220 |
| | ML | 0.003640 | 0.040650 | 0.044290 | 0.000617 | 0.023260 | 0.023877 | 0.000379 | 0.019700 | 0.020079 |
| | OLS | 0.158477 | 0.106145 | 0.264622 | 0.048130 | 0.066590 | 0.114720 | 0.020279 | 0.047890 | 0.068169 |
| | COC | 1.533270 | 0.035150 | 1.568420 | 0.022681 | 0.015500 | 0.038181 | 0.031836 | 0.011431 | 0.043267 |
| 0.8 | HILU | 0.230829 | 0.037640 | 0.268469 | 0.023435 | 0.015511 | 0.038946 | 0.031448 | 0.012832 | 0.044280 |
| | MLGRID | 0.132755 | 0.028100 | 0.160855 | 0.052073 | 0.032270 | 0.084343 | 0.031712 | 0.011580 | 0.043292 |
| | ML | 0.135662 | 0.027930 | 0.1635932 | 0.049880 | 0.015400 | 0.065280 | 0.033562 | 0.011286 | 0.044848 |
| | OLS | 1.023950 | 0.132800 | 1.156750 | 0.369627 | 0.105668 | 0.475295 | 0.176030 | 0.086870 | 0.026290 |
| | COC | 15.797080 | 0.024300 | 15.821383 | 0.096750 | 0.014000 | 0.110750 | 0.044410 | 0.010160 | 0.054570 |
| 0.9 | HILU | 0.294190 | 0.024080 | 0.318270 | 0.102678 | 0.013970 | 0.116648 | 0.040290 | 0.010188 | 0.050478 |
| | MLGRID | 0.763048 | 0.023845 | 0.786893 | 0.367541 | 0.014038 | 0.381579 | 0.246882 | 0.010043 | 0.256925 |
| | ML | 0.721825 | 0.025440 | 0.747265 | 0.357940 | 0.014089 | 0.372029 | 0.256888 | 0.010160 | 0.267048 |

**Table 3:** The Use of RMSE to compare the estimators

| | | T = 20 | | | T = 40 | | | T = 60 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| ρ | ESTIMATORS | $\beta_1$ | $\beta_2$ | SRMSE | $\beta_1$ | $\beta_2$ | SRMSE | $\beta_1$ | $\beta_2$ | SRMSE |
| | OLS | 0.045092 | 0.240308 | 0.285400 | 0.024279 | 0.173061 | 0.197340 | 0.015830 | 0.164770 | 0.180600 |
| | COC | 0.190284 | 0.202684 | 0.392968 | 0.049397 | 0.155595 | 0.240499 | 0.029523 | 0.143000 | 0.172523 |
| 0.4 | HILU | 0.229407 | 0.201478 | 0.430885 | 0.051032 | 0.153549 | 0.204581 | 0.029312 | 0.149916 | 0.179228 |
| | MLGRID | 0.063788 | 0.203507 | 0.267295 | 0.025511 | 0.154306 | 0.179817 | 0.020146 | 0.142371 | 0.162517 |
| | ML | 0.066050 | 0.203630 | 0.269680 | 0.024946 | 0.153258 | 0.178204 | 0.019522 | 0.142064 | 0.161586 |
| | OLS | 0.399388 | 0.326614 | 0.726002 | 0.219784 | 0.259336 | 0.479120 | 0.142424 | 0.221648 | 0.364072 |
| | COC | 1.248322 | 0.187630 | 1.435952 | 0.150602 | 0.124570 | 0.275172 | 0.178439 | 0.107278 | 0.285717 |
| 0.8 | HILU | 0.485267 | 0.194084 | 0.679351 | 0.153114 | 0.124543 | 0.277657 | 0.178411 | 0.113421 | 0.290832 |
| | MLGRID | 0.367160 | 0.167790 | 0.534950 | 0.228201 | 0.180443 | 0.400644 | 0.178347 | 0.107938 | 0.286285 |
| | ML | 0.371061 | 0.167269 | 0.538300 | 0.223310 | 0.124101 | 0.347411 | 0.183505 | 0.106480 | 0.289985 |
| | OLS | 1.016857 | 0.366850 | 1.383707 | 0.608529 | 0.326386 | 0.934915 | 0.420135 | 0.297116 | 0.717251 |
| | COC | 3.993867 | 0.155923 | 4.149790 | 0.312074 | 0.118323 | 0.430397 | 0.211281 | 0.100989 | 0.312270 |
| 0.9 | HILU | 0.542822 | 0.155219 | 0.698041 | 0.322024 | 0.118197 | 0.440221 | 0.201640 | 0.101119 | 0.302759 |
| | MLGRID | 0.883596 | 0.155103 | 1.038699 | 0.606406 | 0.118482 | 0.724888 | 0.497697 | 0.100319 | 0.598016 |
| | ML | 0.861238 | 0.159826 | 1.021064 | 0.598449 | 0.118699 | 0.717148 | 0.507705 | 0.100906 | 0.608611 |

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