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Buys-Ballot estimates for exponential and s-shaped growth curves

I. S. Iwueze
Department of Statistics,
Abia State University, Uturu, Abia State, Nigeria.
and
E. C. Nwogu
Department of Mathematical and Physical Sciences,
Michael Okpara University of Agriculture,
Umudike,, Nigeria.

Abstract.

The Buys-Ballot estimation procedure for time series decomposition when trend-cycle component is either exponential, modified exponential, Gompertz or logistic is discussed in this paper. Estimates are derived for the additive and multiplicative models. A simulated example for the exponential case is used to illustrate the methods developed. Buys-Ballot estimates are compared with the least squares estimates.

Key Words: Buys-Ballot estimates, exponential curve, modified exponential curve, Gompertz curve, logistic curve.

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1.0 Introduction

Any time series data which follow a pattern that repeats after s basic time points within years is said to be seasonal with seasonal period s. For such series, Buys-Ballot (1847) [1] proposed a two-dimensional tabular arrangement (Table 1) with m rows and s columns, according to the number of years/periods and the length of the periodic interval, including the totals and averages.

Table 1: Buys-ballot table for a seasonal time series.

PERIOD	SEASON								TOTAL	AVERAGE
	1	2	..	..	j	..	..	s		
1	X_1	X_2	..	..	X_j	..	..	X_s	$T_{1.}$	$\bar{X}_1$
2	X_{s+1}	X_{s+2}	..	..	X_{s+j}	..	..	X_{2s}	$T_{2.}$	$\bar{X}_2$
..	..	..	..	..	..	..	..	..	..	..
..	..	..	..	..	..	..	..	..	..	..
i	$X_{(i-1)s+1}$	$X_{(i-1)s+2}$	..	..	$X_{(i-1)s+j}$	..	..	X_{is}	$T_{i.}$	$\bar{X}_i$
..	..	..	..	..	..	..	..	..	..	..
..	..	..	..	..	..	..	..	..	..	..
m	$X_{(m-1)s+1}$	$X_{(m-1)s+2}$	..	..	$X_{(m-1)s+j}$	..	..	X_{ms}	$T_{m.}$	$\bar{X}_m$
TOTAL	$T_{.1}$	$T_{.2}$	..	..	$T_{.j}$	..	..	$T_{.s}$	T	
AVERAGE	$\bar{X}_{.1}$	$\bar{X}_{.2}$	..	..	$\bar{X}_{j}$	..	..	$\bar{X}_{.s}$		$\bar{X}_{..}$

where m = number of years/periods
s = length of periodic interval/length of periodicity
$n = ms$ = length of the series

Table 1 displays the within-periods relationships which represent the correlation among observations in the same row (…,X_{t-2}, X_{t-1}, X_t, X_{t+1}, X_{t+2}, ….) and the between-periods relationships which represent the correlation among observations in the same column (…., X_{t-2s}, X_{t-s}, X_{t+s}, X_{t+2s}, ...). The within-periods relationships represent the non-seasonal part while the between-periods relationships represent the seasonal part of the series. Thus, the estimates of the non-seasonal components are derivable from the row averages while the estimates of the seasonal components are derivable from the column averages.

The methods available for analysis of seasonal time series data include the descriptive method and the fitting of probability models (see Chatfield (1980) [2], Wei (1989)). In the descriptive method, the traditional practice is to estimate and isolate the components existing in a study series. Curve-fitting by least squares which is adjudged the most objective method, is used to estimate the trend. The de-trended series is often adopted to estimate the seasonal effects. From the de-trended, de-seasonalized series, the estimates of the cyclical component are obtained by calculating a moving average of the appropriate order.

The whole process of (1) fitting a trend curve by some method and de-trending the series (ii) using the de-trended series to estimate the seasonal indices involved in the traditional method are, no doubt, quite tedious.

Iwueze and Nwogu (2004) [4] developed a new estimation procedure based on the row, column and overall averages of the Buys-Ballot table. The procedure was developed for short period series in which the trend-cycle component (M_t) is jointly estimated and can be represented by a linear equation:

$$M_t = a + bt, \quad t = 1, 2, \ldots, n. \qquad (1.1)$$

Iwueze and Nwogu (2004) [4] gave two alternative methods, namely: (i) the Chain Base Estimation (CBE) method which computes the slope from the relative periodic average changes and (ii) the Fixed Base Estimation (FBE) method which computes the slope using the first period as the base period for the periodic average changes. Estimator of the slope of the line "b" is computed as a weighted average of the periodic average and are given as

$$\hat{b} = \frac{\left(\overline{X}_m - \overline{X}_1\right)}{n-s} \qquad (1.2)$$

for the CBE method and
$$\hat{b} = \left(\frac{1}{n-s}\right)\sum_{i=2}^{m}\left[\left(\frac{X_i - X_1}{i-1}\right)\right] \qquad (1.3)$$

for the FBE method. After obtaining the estimator of the slope, the Buys-Ballot estimator of the intercept is given as
$$\hat{a} = \overline{X} - \hat{b}\left[\frac{(n+1)}{2}\right] \qquad (1.4)$$

Estimators of the parameters of the trend-cycle component are shown to be the same for both the additive and multiplicative models.

Having obtained the Buys-Ballot estimators of the slope and intercept, estimators of the seasonal indices are given by
$$\hat{S}_j = \overline{X}_j - \left\{\overline{X} + \hat{b}\left[\frac{(2j-s-1)}{2}\right]\right\},\ j = 1, 2, \cdots, s \qquad (1.5)$$

for the additive model and
$$\hat{S}_j = \frac{\overline{X}_j}{\left\{\overline{X} + \hat{b}\left[\frac{(2j-s-1)}{2}\right]\right\}},\ j = 1, 2, \cdots, s \qquad (1.6)$$

for the multiplicative model.

Iwueze and Ohakwe (2004) [5] extended the Buys Ballot procedure to the case in which the trend-cycle component is quadratic. That is,

$$M_t = a + bt + ct^2,\ \ t = 1,2,\cdots,n \tag{1.7}$$

In their summary, they have shown that the estimation of slope of the curve is as in Iwueze and Nwogu (2004) [[4]. The difference in method lies in the computation of "c" which is easily computed from differences in the periodic averages. The computation of "c" reduces to

$$\hat{c} = \frac{1}{2(m-2)s}\left[\left(\overline{X}_m - \overline{X}_{(m-1)}\right) - \left(\overline{X}_2 - \overline{X}_1\right)\right] \tag{1.8}$$

for the CBE method and

$$\hat{c} = \frac{1}{2\left(m-2s^2\right)}\left\{\sum_{i=1}^{m-2}\left[\frac{\left(\overline{X}_{(i+2)} - \overline{X}_{(i+1)}\right)}{i}\right] - \overline{X}_2 - \overline{X}_1 \sum_{i=1}^{m-2}\left(\frac{1}{i}\right)\right\} \tag{1.9}$$

for the FBE method. Having estimated "c", estimators of "b" and "a" are given respectively as

$$\hat{b} = \left[\left(\frac{\overline{X}_m - \overline{X}_1}{n-s}\right)\right] - c(\hat{n}+1) \tag{1.10}$$

and

$$\hat{a} = \overline{X} - \frac{\hat{b}(n+1)}{2} - \frac{\hat{c}(n+1)(2n+1)}{6} \tag{1.11}$$

Estimators of the parameters of the trend-cycle component are, here again, shown to be the same for both the additive and multiplicative models. The seasonal indices are, thereafter, given as

$$S_j = \overline{X}_j - d_j \tag{1.12}$$

for the additive model and

$$S_j = \frac{\overline{X}_j}{d_j} \tag{1.13}$$

for the multiplicative model, where for $j = 1, 2, \ldots, s$;

$$d_j = \overline{X} + \frac{\hat{b}(2j-s-1)}{2} - \frac{\hat{c}[(3n-s+1)(s+1) - 6j(n-s+j)]}{6} \tag{1.14}$$

Iwueze and Ohakwe (2004) [5] also showed that the Buys-Ballot estimators for the linear trend-cycle component discussed by Iwueze and Nwogu (2004) [4] are obtainable from the Buys-Ballot estimates of the quadratic trend-cycle component when $c = 0$.

The advantages of the Buys-Ballot procedure is that it computes trend easily and gets over the problem of de-trending a series before computing the estimates of the seasonal effects. However, since the estimators are computed from the means, they may not be reliable when a series contains extreme values (outliers). Furthermore, estimates of the standard errors of the parameter estimates have been provided only empirically; analytical estimators of the standard errors are yet to be obtained. The derivations, when obtained, should be interpreted with caution since normality and independence assumptions are not strictly valid for time series data.

In this paper, the main objective is to obtain the Buys-Ballot estimates of the trend-cycle component and seasonal effects when the trend-cycle component is either exponential (1.15) or modified exponential (1.16).

$$M_t = be^{ct},\ \ t = 1,2,\cdots,n,\ \ c \neq 0 \tag{1.15}$$

or

$$M_t = a + be^{ct},\ \ t = 1,2,\cdots,n,\ \ c \neq 0 \tag{1.16}$$

Observe that when $a = 0$, the modified exponential (1.16) reduces to the exponential curve (1.15). Therefore, we will show only the Buys-Ballot estimates for the modified exponential growth model.

Section 2 presents the procedure for estimation of the parameters. Section 3 contains the numerical examples while Section 4 contains the summary.

2.0 Buys-Ballot Estimates.

This Section considers the Buys-Ballot parameter estimation procedure when the trend-cycle component is the modified exponential equation given by (1.16). Section 2.1 discusses the procedure for the additive model while the procedure for the multiplicative model is the object of Section 2.2. Section 2.3 gives simple applications to the exponential, Gompertz and logistic curves.

2.1 Buys-Ballot procedure for the Additive Model.

The additive model is given by

$$X_t = M_t + S_t + e_t, \quad t = 1,2,\cdots,n, \qquad (2.1)$$

where for time t, X_t is the value of the series, S_t is the seasonal component whose sum over a complete period is zero, e_t is the irregular component which, for our discussion, is the Gaussian $N(0,\sigma^2)$ white noise and M_t is the modified exponential trend-cycle component given in Equation (1.16). Methods of obtaining the row, column and overall totals and averages are those of Iwueze and Nwogu (2004) [4] and Iwueze and Ohakwe (2004) [5]. Only the results are given

$$T_i = sa + b\left(\frac{1-e^{cs}}{1-e^{c}}\right) e^{c[(i-1)s+1]}, \; i = 1,2,\cdots,m \qquad (2.2)$$

$$\bar{X}_i = a + \left(\frac{b\left(1-e^{cs}\right)}{s\left(1-e^{c}\right)}\right) e^{c[(i-1)s+1]}, \; i = 1,2,\cdots,m \qquad (2.3)$$

$$T_j = ma + b\left(\frac{b\left(1-e^{cn}\right)}{s\left(1-e^{cs}\right)}\right) e^{cj} + mS_j, \; j = 1,2,\cdots,s \qquad (2.4)$$

$$\bar{X}_i = a + \left(\frac{b\left(1-e^{cn}\right)}{m\left(1-e^{cs}\right)}\right) e^{cj} + S_j, \; j = 1,2,\cdots,s \qquad (2.5)$$

$$T = na + \left(\frac{be^{c}\left(1-e^{cn}\right)}{\left(1-e^{c}\right)}\right) \qquad (2.6)$$

and

$$\bar{X}_{..} = a + \left(\frac{be^{c}\left(1-e^{cn}\right)}{n\left(1-e^{c}\right)}\right) \qquad (2.7)$$

2.1.1 *Estimates of a, b and c.*

The row averages given in (2.3) are functions of "a", "b" and "c" only. Let Y_i be the first order differences of the row averages. Then

$$Y_i = \nabla\bar{X}_i = \bar{X}_{(i+1)} - \bar{X}_i, \; i = 1,2,\cdots,m-1 \qquad (2.8)$$

$$= \left(\frac{b\left(1-e^{cs}\right)^2}{s\left(1-e^{-c}\right)}\right) e^{c\,i(i-1)\,s} \qquad (2.9)$$

Thus, the first order difference (Y_i) of the periodic averages are functions of b and c only. The ratio of Y_{i+1} and Y_i is given by

$$\frac{Y_{i+1}}{Y_i} = e^{cs}, \; i = 1,2,\cdots,m-2 \qquad (2.10)$$

Alternatively, the ratio of Y_{i+1} and Y_1 is given by

$$\frac{Y_{i+1}}{Y_i} = e^{cis}, \; i = 1,2,\cdots,m-2 \qquad (2.11)$$

Let $$Z_i = \ln Y_{i+1} - \ln Y_i = cs, \; i = 1,2,\cdots,m-2 \qquad (2.12)$$

And $$W_i = \ln Y_{i+1} - \ln Y_i = cis, \; i = 1,2,\cdots,m-2 \qquad (2.13)$$

Then from (2.12), $$\hat{c}_i^{(1)} = \frac{Z_i}{s}, \; i = 1,2,\cdots,m-2 \qquad (2.14)$$

and from (2.13) $$\hat{c}_i^{(2)} = \frac{W_i}{is}, \; i = 1,2,\cdots,m-2 \qquad (2.15)$$

Thus, the computation of "c" from (2.14) or (2.15) is done by expressing the changes in the logarithm of the first order differences of the periodic averages as differences with reference to the logarithm of the differenced average values at some earlier period. As noted by Iwueze and Nwogu (2004) [4], two alternatives are possible: (i) Equation (2.14) computes "c" using the relative differences of the logarithm of the first order differences of the periodic averages (Chain Base estimation (CBE) method); (ii) Equation (2.15) computes "c" using the logarithm of the first differences (Y_1) of the first order differences as the earlier period (Fixed Base Estimation (FBE) method). The two possibilities will each give rise to (*m* - 2) different estimates of "c". The average of these (*m* - 2) different estimates will be taken as the Buys-Ballot estimate of "c" and their associated standard error as the standard error of the estimate.

Having obtained the estimate of "c" we use (2.9) to find (*m* - 1) different estimates of "b". Again, the average of these (*m* - 1) different values will be taken as the Buys-Ballot estimate of "b" and their associated standard error as the standard error of the estimate. Of course, from the summation of (2.8) and (2.9), we obtain

$$\hat{b} = \frac{s\left(1-c^{-\hat{c}}\right)\left[\bar{X}_m - \bar{X}_i\right]W_i}{\left(1-e^{\hat{c}s}\right)\left(1-e^{\hat{c}(n-s)}\right)} \qquad (2.16)$$

After estimating "c" and "b", we use (2.3) to find m different estimates of "a". Here again, the average of these *m* different estimates of "a" will be taken as the Buys-Ballot estimate of "a". From (2.3)

$$\hat{a}_i = \bar{X}_i - \left(\frac{b\left(1-e^{\hat{c}s}\right)}{s\left(1-e^{\hat{c}}\right)}\right) e^{\hat{c}[(i-1)s+1]}, \; i = 1,2,\cdots,m \qquad (2.17)$$

$$\hat{a} = \frac{1}{m}\sum_{i=1}^{m}\hat{a}_i \qquad (2.18)$$

$$= \bar{X} + \left(\frac{b\left(1-e^{n\hat{c}}\right)}{n\left(1-e^{\hat{c}}\right)}\right) \qquad (2.19)$$

This result (2.19) could also have been derived from the expression for the overall average in (2.7). When there is no trend (exponential growth with $c = 0$) and $b = 0$, we obtain

$$\hat{a} = \bar{X}_{..} \qquad (2.20)$$

2.1.2 *Estimates of* $S_j, \; j=1,2,\cdots,s$

The estimates of the seasonal effects are obtained by substituting the estimates of "a", "b" and "c" into the expression for the column averages given in (2.5). Hence,

$$\hat{S}_j = \bar{X}_j - \delta_j \qquad (2.21)$$

where, $$\delta_i = \hat{a}_i + \left(\frac{b\left(1-e^{\hat{c}s}\right)}{m\left(1-e^{\hat{c}s}\right)} \right) e^{\hat{c}j}, \quad j = 1,2,\cdots,s \tag{2.22}$$

When there is no trend (exponential growth) and $b = 0$, it is clear from (2.22) that

$$\hat{S}_j = \overline{X}_j - \overline{X}_{..} \tag{2.23}$$

2.2 Buys-Ballot procedure for the Multiplicative Model

The multiplicative model is given by $X_t = M_t \cdot S_t \cdot e_t,\ t = 1,2,\cdots,n$ (2.24)

where for time t, X_t is the value of the series, S_t is the seasonal component whose sum over a complete period is s; e_t is the irregular component which, for our discussion, is the Gaussian $N(1,\sigma_1^2)$ white noise and M_t is the modified exponential trend-cycle component given in Equation (1.16).

The row, and overall totals and averages are the same as those obtained for the additive model (2.1) given in Equations (2.2), (2.3), (2.6) and (2.7). On the other hand, the column totals and averages are:

$$T_j = \left\{ ma + b\left(\frac{\left(1-e^{cn}\right)}{\left(1-e^{cs}\right)} \right) e^{cj} \right\} S_j, \quad j = 1,2,\cdots,s \tag{2.25}$$

$$\overline{X}_j = \left\{ a + \left(\frac{b\left(1-e^{cn}\right)}{m\left(1-e^{cs}\right)} \right) e^{cj} \right\} S_j, \quad j = 1,2,\cdots,s \tag{2.26}$$

2.2.1 *Estimates of a, b and c.*

Since the row and overall averages are the same for the additive and multiplicative models, the Buys-Ballot estimates of the parameters of the trend-cycle component are the same for both models.

2.2.2 *Estimates of* $S_j,\ j=1,2,\cdots,s$

The estimates of the seasonal effects are obtained by substituting the estimates of "a", "b" and "c" into the expression for the column averages given in (2.26). Hence,

$$\hat{S}_j = \frac{\overline{X}_j}{\delta_j} \tag{2.27}$$

This reduces to $$\hat{S}_j = \frac{\overline{X}_j}{\overline{X}} \tag{2.28}$$

when there is no exponential growth and $b = 0$.

2.3 Important Remark.

2.3.1 *Exponential growth curve*.

When a = 0, the modified exponential (1.16) reduces to the exponential curve (1.15). To obtain the (m - 1) estimates of "c", we use the periodic averages instead of the first order differences of the periodic averages. That is, Z_i and W_i are now given by

$$Z_i = \ln \overline{X}_{i+1} - \ln \overline{X}_i = cs, \quad i = 1,2,\cdots,m-1 \tag{2.29}$$

and $$W_i = \ln \overline{X}_{i+1} - \ln \overline{X}_i = cis, \quad i = 1,2,\cdots,m-1 \tag{2.30}$$

The m estimates of "b" are given by $$\hat{b} = \left(\frac{s\left(1-e^{\hat{c}}\right)}{e^{\hat{c}}\left(1-e^{\hat{c}s}\right)} \right) \overline{X}_i e^{-\hat{c}(i-1)s}, \quad i = 1,2,\cdots,m \tag{2.31}$$

and $$\hat{b} = \left(\frac{n\left(1-e^{\hat{c}}\right)}{e^{\hat{c}}\left(1-e^{\hat{c}s}\right)} \right) \overline{X} \tag{2.32}$$

2.3.2 ***Gompertz and Logistic growth curves***.

The Gompertz curve is given by

$$\log X_t = a - be^{ct}, \quad t = 1,2,\cdots,n \tag{2.33}$$

where a, b, c are parameters with $c < 0$, while the logistic curve is given by

$$X_t = \frac{a}{\left[1+be^{ct}\right]}, \; t = 1,2,\cdots,n \tag{2.34}$$

or

$$\left(\frac{1}{X_t}\right) = \left(\frac{1}{a}\right) + \left(\frac{b}{a}\right)e^{ct}, \; t = 1,2,\cdots,n \tag{2.35}$$

Both these curves are S-shaped and approach an asymptotic value as $t \to \infty$.

For all curves of this type, the fitted function provides a measure of the trend, but fitting the curves to data may lead to non-linear simultaneous equations (see Levenbach and Reuter (1976) [6], Harrison and Pearce (1972) [3]). Upon comparing the form of (2.33) and (2.35) with (1.16), without X_t, for the special but important cases of Gompertz and logistic growth curves, we find that the Buys-Ballot estimates for the modified exponential growth curve can be generalized to obtain the Buys-Ballot estimates for the Gompertz and Logistic growth curves.

3.0 Empirical example

In this Section, we use the multiplicative model of the exponential growth curve to illustrate the methods described in Section 2. Our example shows a simulation of 100 values from the model

$$X_t = \left(be^{ct}\right)S_t e_t \tag{3.1}$$

with $s = 4$, $b = 10.0$, $c = 0.02$, $S_1 = 0.6$, $S_2 = 1.1$, $S_3 = 0.9$, $S_4 = 1.4$ and e_t being Gaussian N(1.0, 0.0625) white noise. The Buys-Ballot table for the series is given in Table 2. As shown in Table 2 and Figure 1, it is clearly seasonal with an exponential trend.

First, the parameters of the trend-cycle component (listed in Table 5) were obtained by least squares estimation (LSE) procedure, using MINITAB. The seasonal indices (also listed in Table 5) were obtained by averaging the de-trended series. The residuals indicate no model inadequacy with respect to the autocorrelation function (ac.f). The residual mean and residual standard deviation obtained from the LSE procedure are also listed in Table 5.

The computational procedure for the Buys-Ballot estimates for the exponential trend curve is laid out in Table 3, while the computational procedure for estimating the seasonal indices is laid out in Table 4. The CBE and FBE estimates can each be used to obtain component analysis tables after which the residuals obtained can then be checked for randomness. For both methods, the residual ac.f's indicate no inadequacy.

Table 2: Simulated data from $X_t = (be^{ct})$, S_t, e_t. with $s = 4$, $b = 10.0$, $c = 0.02$, $S_1 = 0.6$, $S_2 = 1.1$, $S_3 = 0.9$, $S_4 = 1.4$, e_t, N(1.0, 0.0625).

PERIOD	SEASON				TOTAL	AVERAGE	STD
	I	II	III	IV			
1	7.586	11.061	11.319	11.057	41.023	10.2558	1.7840
2	4.815	14.229	14.719	6.667	40.430	10.1075	5.1023
3	6.163	16.699	13.173	19.646	55.681	13.9203	5.8092
4	6.145	13.212	7.578	21.198	48.133	12.0332	6.8289
5	10.345	17.997	9.915	28.805	67.062	16.7655	8.8434
6	11.258	12.479	17.993	23.464	65.194	16.2985	5.6039
7	10.875	18.997	16.580	23.722	70.174	17.5435	5.3443
8	13.922	19.705	23.058	19.853	76.538	19.1345	3.8038

PERIOD	SEASON				TOTAL	AVERAGE	STD
	I	II	III	IV			
9	11.869	24.868	26.441	30.273	93.451	23.3627	7.9916
10	5.833	21.714	15.258	30.037	72.842	18.2105	10.2317
11	12.353	23.807	24.742	34.438	95.340	23.8350	9.0385
12	15.765	25.309	24.205	49.761	115.040	28.7600	14.6352
13	14.804	41.846	24.373	42.176	123.199	30.7998	13.5229
14	12.477	40.366	39.293	40.714	132.850	33.2125	13.8369
15	20.239	40.849	25.746	40.470	127.304	31.8260	10.4460
16	18.159	36.466	32.110	42.898	129.633	32.4083	10.4822
17	23.566	43.408	27.569	38.966	133.509	33.3773	9.3425
18	29.257	35.421	26.544	47.209	138.431	34.6077	9.1853
19	21.621	60.949	52.298	61.777	196.645	49.1612	18.8539
20	35.906	55.518	55.073	40.205	186.702	46.6755	10.1087
21	43.245	72.032	32.360	76.159	223.796	55.9490	21.4860
22	19.224	29.537	59.193	76.393	184.347	46.0867	26.3676
23	19.460	86.480	28.493	101.046	235.479	58.8697	40.8943
24	41.663	74.141	62.105	98.534	276.443	69.1107	23.7591
25	60.426	64.126	49.145	40.918	214.615	53.6538	10.6154
TOTAL	476.976	901.216	719.283	1046.386	3143.861		
AVERAGE	19.0790	36.0486	28.7713	41.8554		31.4386	
STD	13.6410	21.6798	15.7313	24.4498			20.9071

STD = Standard deviation.

Table 4: Buys-Ballot estimates of the seasonal indices (multiplicative model with exponential trend).

j	$\bar{X}_j$	CBE			FBE		
		δ_j	$\hat{S}_j = \frac{\bar{X}_j}{\delta_j}$	Adjusted $\hat{S}_j$	δ_j	$\hat{S}_j = \frac{\bar{X}_j}{\delta_j}$	Adjusted $\hat{S}_j$
1	19.0790	30.1197	0.6334	0.6252	31.4716	0.6062	0.6288
2	36.0486	30.6435	1.1764	1.1612	32.1241	1.1222	1.1640
3	28.7713	31.1764	0.9229	0.9110	32.7901	0.8774	0.9101
4	41.8554	31.7185	1.3196	1.3026	33.4699	1.2505	1.2971

Table 3: Buys-Ballot estimates for the parameters of the exponential trend curve

I	$\bar{X}_i$	CBE			FBE		
		Z_i	$\hat{c}_i^{(1)}$	$\hat{b}_i^{(1)}$	W_i	$\hat{c}_i^{(2)}$	$\hat{b}_i^{(2)}$
1	10.2558	-0.01457	-0.00364	9.8213	-0.01457	-0.00364	9.7403
2	10.1075	0.32007	0.08002	9.0343	0.30550	0.03819	8.8431
3	13.9203	-0.14567	-0.03642	11.6132	0.15983	0.01332	11.2191
4	12.0332	0.33165	0.08391	9.3700	0.49148	0.03072	8.9341
5	16.7655	-0.02825	-0.00706	12.1849	0.46323	0.02316	11.4666
6	16.2985	0.07361	0.01840	11.0562	0.53684	0.02237	10.2688
7	17.5435	0.08681	0.02170	11.1077	0.62365	0.02227	10.1822
8	19.1345	0.19965	0.04991	11.3077	0.82330	0.02573	10.2305
9	23.3627	-0.24914	-0.06229	12.8865	0.57416	0.01595	11.5068
10	18.2105	0.26916	0.06729	9.3752	0.84331	0.02108	8.2624

I	$\overline{X}_i$	CBE			FBE		
		Z_i	$\hat{c}_i^{(1)}$	$\hat{b}_i^{(1)}$	W_i	$\hat{c}_i^{(2)}$	$\hat{b}_i^{(2)}$
11	23.8350	0.18783	0.04696	11.4532	1.03114	0.02344	9.9621
12	28.7600	0.06852	0.01713	12.8989	1.09966	0.02291	11.0734
13	30.7998	0.07542	0.01885	12.8932	1.17508	0.02260	10.9242
14	33.2125	-0.04265	-0.01066	12.9768	1.13244	0.02022	10.8517
15	31.8260	0.01812	0.00453	11.6064	1.15057	0.01918	9.5793
16	32.4083	0.02946	0.00737	11.0312	1.18003	0.01844	8.9858
17	33.3773	0.03621	0.00905	10.6040	1.21624	0.01789	8.5252
18	34.6077	0.35103	0.08776	10.2623	1.56726	0.02177	8.1430
19	49.1612	-0.05188	-0.01297	13.6064	1.51538	0.01994	10.6558
20	46.6755	0.18122	0.04531	12.0576	1.69660	0.02121	9.3198
21	55.9490	-0.19391	-0.04848	13.4901	1.50268	0.01789	10.2911
22	46.0867	0.24480	0.06120	10.3717	1.74748	0.01986	7.8090
23	58.8697	0.16038	0.04010	12.3656	1.90787	0.02074	9.1890
24	69.1107	-0.25316	-0.06329	13.5494	1.65471	0.01724	9.9374
25	53.6538	——	——	9.8181	——	——	7.1069
TOTAL			0.41369	286.7419		0.49248	243.0076
AVERAGE			0.01724	11.4697		0.02052	9.7203
STD			0.04374	1.4013		0.00712	1.1910

It is clear from Table 5 that the FBE method will be preferred over both the LSE and CBE methods when the error mean and error variance are used as basis for comparison.

4.0 Conclusion

By deriving the Buys-Ballot estimates for the parameters of the trend equation and seasonal indices of the modified exponential growth curve, we have shown that a simple adaptation of our procedure will give the Buys-Ballot estimates of the parameters of the trend equation and seasonal indices of the exponential, Gompertz and logistic growth curves.

Table 5: Summary of estimates (Multiplicative model with exponential trend).

PARAMETER	ACTUAL VALUE	LSE	CBE	FBE
b	10.00	9.6796 ± 0.7943	11.4692 ± 1.4013	9.7203 ± 1.1910
c	0.02	0.0190 ± 0.0014	0.0172 ± 0.0437	0.0205 ± 0.0071
S_1	0.60	0.6056	0.6252	0.6288
S_2	1.10	1.1483	1.1612	1.1640
S_3	0.90	0.9365	0.9110	0.9101
S_4	1.10	1.3096	1.3026	1.2971
Error Mean	1.00	1.0772	0.9934	0.9925
Error STD (σ_1)	0.25	0.2652	0.2527	0.2482

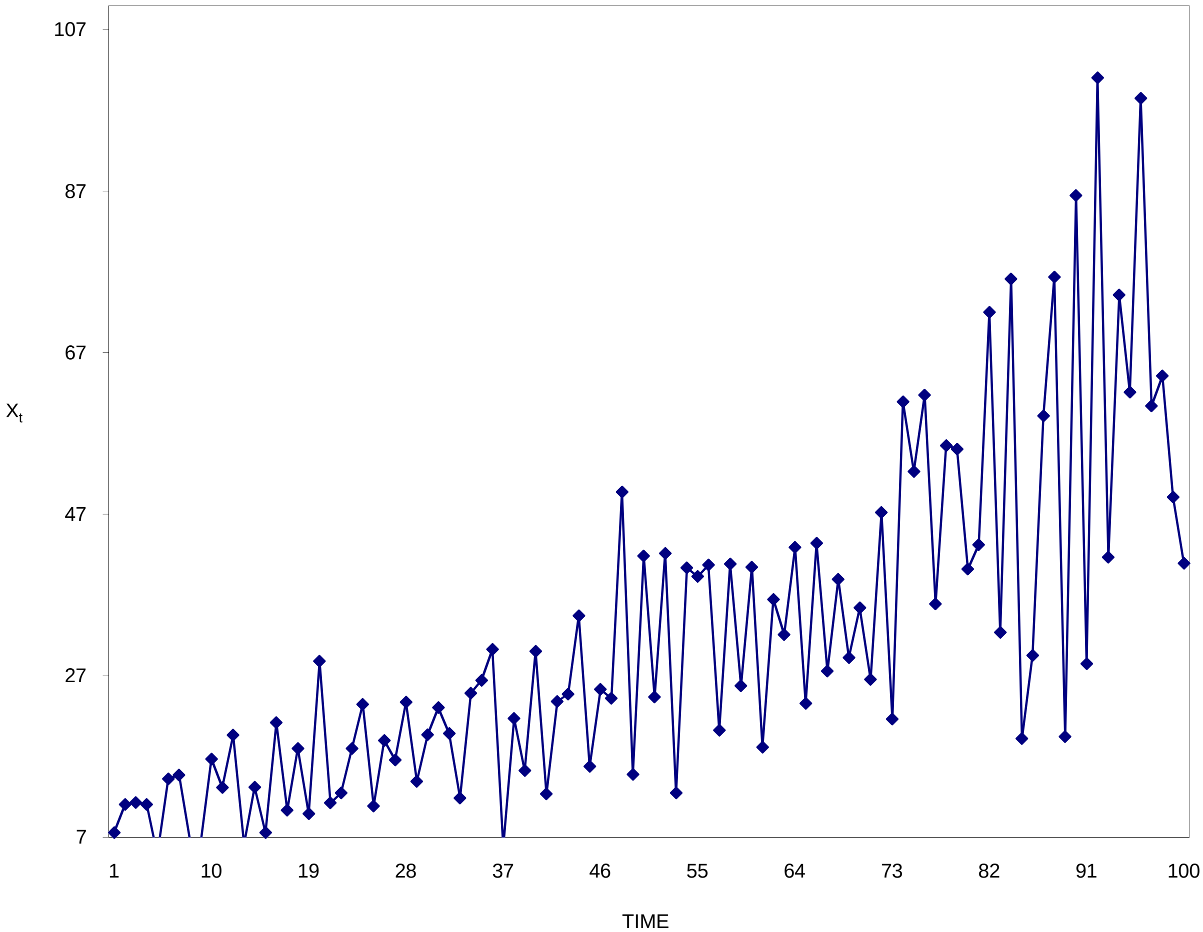

Figure 1: A simulated multiplicative series $X_t = (be^{ct}), S_t e_t$ with s = 4, b = 10.0, c = 0.02, S_1 = 0.6, S_2 = 1.1, S_3 = 0.9, S_4 = 1.4, $e_t \sim N(1.0, 0.0625)$.

References

[1] Buys-Ballot, C. H. D., 1847, Leo Claemert Periodiques de Temperature, Kemint et Fils, Utrecht.

[2] Chatfield, C., 1980, The Analysis of Time Series: An Introduction, Chapman and Hall, London.

[3] Harrison, P. J. and Pearce, S. F., 1972, The use of trend curves as an aid to market forecasting, Industrial Marketing Management, 2, pp. 149-170.

[4] Iwueze, I. S. and Nwogu, E. C., 2004, Buys-Ballot estimates for time series decomposition. Global Journal of Mathematical Sciences, Vol. 3, No. 2, pp. 83-98.

[5] Iwueze, I. S. and Ohakwe Johnson, 2004, Buys-Ballot estimates when stochastic trend is quadratic. Journal of the Nigerian Association of Mathematical Physics, Vol. 8, pp. 311-318.

[6] Levenbach, H. and Reuter, B. E., 1976, Forecasting trending time series with relative Growth rate models, Technometrics, 18, pp. 261-272.