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Comment on computation of oscillating integrals

V. U. Aihie
Department of Mathematics
University of Benin
Benin City

Abstract

The arbitrary nature of the terminating condition in an existing routine in the 'computation of oscillating integrals' [7] ***is examined and changed. In fact the terminating condition is made integrand dependent. The new routine is demonstrated to show that it is a generalization of the existing routine.***

Keywords: Oscillating integrals, quadrature, chebyshev polynomial, automatic computation.

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1.0 Introduction

Robert Piessens and Maria Branders have described their routine in [7] for adaptive quadrature for automatic computation of oscillating integrals. The steps taken by the two authors are as follow:

(i) They considered the integrations of both

$$s(w) = \int_a^b \sin wx f(x) dx \tag{1.1}$$

and

$$c(w) = \int_a^b \cos wx f(x) dx \tag{1.2}$$

with a user – specified tolerance $\in_a$ or relative tolerance $\in_r$.

(ii) With the main procedure for evaluating

$$s_{1_j}(w) \approx \int_{I_j} \sin w(x) f(x) dx \tag{1.3}$$

(iii) a method for calculating $\in_{I_j}(w)$ an estimate for the error

$$e_j = \left| s_{I_j}(w) - \int_{I_j} \sin wx\ f(x) dx \right| \tag{1.4}$$

so that the formal algorithm is then

(a) Let

$$\text{I = [a, b]} \tag{1.5}$$

Calculate $S_I(w)$ and $e_I(w)$ if

$$e_I(w) \le \max\{a_a, e_I \left| s_I(w) \right|\} \tag{1.6}$$

The computation is terminated and $s_I(w)$ is returned as the approximate value of $s(w)$ otherwise the interval is divided into two subintervals I_1 and I_2.

(b) At step n of the algorithm $(n = 1, 2, ---n)$ the interval is divided into n subintervals

$I_j \; j = 1, 2 --- n$ for each of the values $s_{I_j}(w)$ and $e_{I_j}(w)$ are known from previous steps or must be calculated.

Set

$$\sigma_n = \sum_{j=1}^{n} s_{I_j}(w) \tag{1.7}$$

and

$$\in_n = \sum_{j=1}^{n} e_{I_j}(w) \tag{1.8}$$

if

$$\in_n = \max\{\in_a, \in_r |6_n|\} \tag{1.9}$$

σ_n is returned as an approximate value of $s(w)$ and the integration is terminated.

Otherwise, the interval I_k on which

$$e_{I_k}(w) = \max\{e_{I_j}(w), j = 1, 2, --- n\} \tag{1.10}$$

is divided into two equal subintervals. We have next $n + 1$ subintervals and this leads to steps $n + 1$ of the algorithm. Details of the method of evaluating the integral is as follows

Let $I_j = [\alpha, \beta]$ an arbitrary subinterval [7]. Then

$$\int_{I_j} \sin wx \, f(x)dx = c_1\left\{\cos(c_1 w)\int_{-1}^{1}\sin(\mu t)\phi(t)dt + \sin(c_1 w)\int_{-1}^{1}\cos(\mu t)\phi(t)dt\right\} \tag{1.11}$$

where

$$c_1 = (\beta - \alpha)/2 \tag{1.11a}$$

$$c_2 = (\beta + \alpha)/2 \tag{1.11b}$$

$$\mu = c_1 w \tag{1.11c}$$

and

$$\phi(t) = f(c_2 + c_1 t) \tag{1.11d}$$

Furthermore they considered the truncated chebyshev series approximation

$$\phi(t) = \sum_{t=0}^{12}{}' a_i T_i(t) \tag{1.12}$$

with the single prime denoting that the first term is taken with the factor ½ and where

$$a_i = \frac{1}{6}\sum_{j=0}^{12}{}^{//}\phi\left(\frac{\cos\pi j}{12}\right)\frac{\cos\pi ji}{12} \qquad (1.13)$$

where the double prime indicates that both the 1st and last term are taken with a factor ½ [3, 7]. Putting (1.13) in (1.12) results in equation (1.14)

$$S_{I_j}(w) = c_{I_j}\sum_{j=0}^{12} a_j\left(\cos(c_2 w)s_j + \sin(c_2 w)c_j\right) \qquad (1.14)$$

where

$$S_j = \int_{-1}^{1}\sin(\mu t)T_j(t)dt \qquad (1.15)$$

and

$$c_j = \int_{-1}^{1}\cos(\mu t)T_j(t)dt \qquad (1.16)$$

These integrals are computed using the three term recurrence relations

$$\mu^2(j-1)(j-2)c_{j+2} - 2\left(j^2-4\right)\left(\mu^2 - 2j^2 \_ 2\right)c_j + \mu^2(j+1)(j+2)c_{j-2} = 24\mu\sin\mu - 8\left(j^2-4\right)\cos\mu \qquad (1.17)$$

and

$$\mu^2(j-1)(j-2)s_{j+2} - 2\left(j^2-4\right)\left(\mu^2 - 2j^2 \_ 2\right)s_j + \mu^2(j+1)(j+2)s_{j-2} = -8\left(j^2-4\right)\sin\mu - 24\mu\cos\mu \qquad (1.18)$$

The totality of the above leads to the subroutine AINOS which also calls another subroutine AICHMO. The details above are the work of Piessens and Branders [7].

2.0 Observation and Modification

The kernel of the work in the computation of oscillating integrals' [7] is the approximation of a function by the truncated chebyshev series as shown in the equation (1.12), that is,. $\phi(t) \approx \sum_{t=0}^{12} a_i T_i(t)$.

As it can be noticed in the final computation, the above approximation runs through the whole work. So the impression is created that all oscillating functions can be approximated accurately by the sum $\sum_{i=0}^{12}{}^{/} a_i T_i(t)$. This is arbitrary and from various works in [1], [2], [3] the presumption cannot be correct and in fact it is not correct. It is as a result of this we are suggesting that the number of terms should be determined by the level of error that may be tolerated in any particular integration.

Therefore, any arbitrary function $\phi(t)$ may be approximated as

$$\phi(t) \approx \sum_{i=0}^{n}{}^{/} a_i T_i(t) \qquad (2.1)$$

where

$$\left|\sum_{i=0}^{n+1}{}^{/} a_i T_i(t) - \sum_{i=0}^{n}{}^{/} a_i T_i(t)\right| <\in_t \qquad (2.2)$$

where $\in_t$ is a defined level of error tolerance. In this way N^* (the number of terms of chebyshev polynomial) will depend on each function in question. So substituting this is in equation (1.13) we have

$$a_i = \frac{2}{N^*}\sum_{j=0}^{N^*}{}^{//}\phi\left(\cos\frac{\pi j}{N^*}\right)\cos\frac{\pi ji}{N^*} \qquad (2.3)$$

In a similar manner using the same argument in [7] we obtain

$$S_{I_j}(w) = c_j \sum_{j=0}^{N^*}{}^{//} a_j\left(\cos(c_2 w)s_j + \sin(c_2 w)c_j\right) \qquad (2.4)$$

where

$$S_j = \int_{-1}^{1}\sin(\mu t)T_j(t)dt \qquad (2.5)$$

and

$$c_j = \int_{-1}^{1}\cos(\mu t)T_j(t)dt \qquad (2.6)$$

These integral (2.5) and (2.6) can be computed using the three term recurrence relations

$$\mu^2(j-1)(j-2)c_{j+2} - 2\left(j^2-4\right)\left(\mu^2 - 2j^2 + 2\right)c_j + \mu^2(j+1)(j+2)c_{j-2} = 24\mu\sin\mu - 8\left(j^2-4\right)\cos\mu t \qquad (2.7)$$

and

$$\mu^2(j-1)(j-2)s_{j+2} - 2\left(j^2-4\right)\left(\mu^2 - 2j^2 \_ 2\right)s_j + \mu^2(j+1)(j+2)s_{j-2} = -8\left(j^2-4\right)\sin\mu - 24\mu\cos\mu \qquad (2.8)$$

where according to Piessens at el [7]

$$c_0 = 2\frac{\sin\mu}{\mu} \qquad (2.9)$$

$$c_2 = 8\frac{\cos\mu}{\mu^2} + \left(2\mu^2 - 8\right)\sin\frac{\mu}{\mu^3} \qquad (2.10)$$

$$s_1 = \frac{2\left(\sin\mu - \mu\cos\mu\right)}{\mu^2} \qquad (2.11)$$

$$s_3 = \sin\mu \frac{\left(\frac{18-48}{\mu^2}\right)}{\mu^2} + \cos\mu \frac{\left(\frac{48}{\mu^2 - 2}\right)}{\mu^2} \qquad (2.12)$$

and

$$c_{2i+1} = s_{2i} = 0 \quad i = 0,\ 1,\ 2, \cdots \qquad (2.13)$$

and the estimate of the integration error is

$$e_{I_j}(w) = |c_1| \sum_{j=N-3}^{N^*} |a_j| \left(|\cos(c_2 w) s_j| + |\sin(c_2 w)\, c_j| \right) \qquad (2.14)$$

Apart from this modification, all other steps are the same as in the computation of oscillating integrals [7]. The subroutine AICHMO (which is called by AINOS) is modified by the terminating criterion in (2.11).

3.0 Experimental Computation and Result

The result of the experiment was compared with the subroutine AINOS (with AICHMO) [5] and the subroutine $FSPL_2$ [3, 4, 6] which is a standard robust algorithm. Table 1 gives comparative results for the integral

$$\int_0^1 \frac{\cos 2n\pi}{1 + 2\alpha \cos nx + \alpha^2} dx = \frac{(-1)^n \alpha^n}{1 + \alpha^n},\ |\alpha| < 1.$$

For various values of n and $\varepsilon = \varepsilon_a = \varepsilon_1$

Note if $|\alpha|$ is small the integral $\frac{\cos 2n\pi x}{1 + 2\alpha \cos \pi x + \alpha^2}$ is a smooth function while if $\alpha \approx 1$ it is peaked and so has a singularity on [0, 1]. In the table N denote the number of function evaluation and T represent the computation time in – seconds.

Table 1

ε	α	n	NEW SCHEME			AINOS			FSPL2		
			Absolute error	**N**	**T**	**Absolute error**	**N**	**T**	**Absolute error**	**N**	**T**
10^{-6}	0.2	2	-0.26×10^{-8}	39	23	-0.26×10^{-8}	39	23	0.26×10^{-8}	65	30
		8	0.14×10^{-8}	91	36	0.14×10^{-8}	91	36	-0.16×10^{-8}	65	36
	0.9	2	0.13×10^{-10}	273	143	0.13×10^{-10}	273	143	-0.21×10^{-8}	513	220
		8	-0.15×10^{-11}	299	146	-0.15×10^{-11}	299	146	0.22×10^{-8}	513	210
		32	-0.42×10^{-8}	273	119	-0.42×10^{-8}	273	119	0.11×10^{-7}	1025	476
	0.2	2	0.80×10^{-14}	143	77	0.80×10^{-14}	143	77	0.62×10^{-12}	257	130
		8	0.14×10^{-14}	91	37	0.14×10^{-14}	91	37	-0.56×10^{-11}	257	130

10^{-9}	0.9	2 8 32	0.77×10^{-12} -0.15×10^{-11} -0.53×10^{-12}	50 7 42 9 48 1	27 6 22 0 19 2	0.77×10^{-12} -0.15×10^{-11} -0.53×10^{-12}	50 7 42 9 48 1	27 6 22 0 19 2	-0.18×10^{-12} 0.33×10^{-12} -0.38×10^{-13}	1025 2049 4097	453 879 1724

4.0 Conclusion

It was found that the experimental results were the same as for AINOS (with AICHMO) and that the new routine competes favourably with FSPL2 for smooth integrals. $\alpha \cong 1$. It is more accurate than AINOS and FSPL2. In fact the routine AINOS is a particular form of the new method or rather the new method is a generalization of the routine AINOS.

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